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STUDENTS' ACTIVITIES ABOUT FUNCTIONS AT UPPER SECONDARY LEVEL: A GRID FOR DESIGNING A DIGITAL ENVIRONMENT AND ANALYSING USES

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We present a grid that organises and connects various students' activities about functions at upper secondary level. We built this grid to overcome what we saw as a fragmentation of research, and to work on the design and experimentation of one of the digital environments of the ReMath European project, Casyopée. Analysing classroom use of the environment, we show how the grid helps to make sense of its potentialities, especially as a tool for functional modeling. Finally, we situate this outcome inside the theoretical ReMath work that aims to progress in connecting and integrating theoretical frames in technology enhanced mathematics learning.

INTRODUCTION

This paper deals with students' activities related to the concept of function at upper secondary level. We are particularly attentive to how in many curricula like in France, activities in the functional world are supposed to engage and support the transition between algebra and calculus and how digital technologies can enrich students' activities in that area and make these more productive.

As pointed out by Kieran (2007) in her recent review of the literature on learning and teaching algebra, particular emphasis has been put on the entrance in the algebraic world at middle school level or even earlier. Nevertheless, a substantial body of research also exists at upper secondary level and pays special attention to digital technologies, some giving more emphasis to activities supported by Computer Algebra Systems (CAS) and the role they can play for supporting the understanding of algebraic equivalence and forms (Kieran, Drijvers 2006), and others stressing the role that activities based upon the use of varied computer representations can play in the understanding of functional dependencies (Arzarello, Robutti 2004, Falcade et al. 2007) or in the introduction to calculus concepts (Maschietto 2008).

Our concern is that, at the moment, these approaches are rather unconnected, and thus insufficient for analyzing classroom activities, relatively to the global challenge of learning about functions at upper secondary level. Our experience deals with the design and experiment of six digital didactic artefacts in the Remath European project, one of these, Casyopée, being developed by our team to specifically address this challenge (Lagrange, Gélis 2008). In this research, the fragmentation of frameworks has been pointed out as a difficulty when trying to make sense of their didactical functionalities (Artigue, Cerulli 2008).

In this paper, we present a grid that we think useful to connect various activities about functions. We show how it helps to make sense of Casyopée's features especially as a help for functional modeling and for connecting varied activities. Finally, we study an example of students' activities using Casyopée and we situate this grid inside the theoretical ReMath work.

THE GRID

		Representations and Types of activities				
		Enactive-Iconic		Algebraic		
		Local	Global	Generational	Transforma.	GlobalMeta
Objects repre-sented	Covariation and dependency in a physical system Covariation and dependency between magnitudes or measures Mathemati-cal Functions of one real variable	Small moves. Obser-ving effect on elements Small moves. Obser-ving effect on values Tracing graphs Browsing Tables	Moving elements Obser-ving transfor-mations Graphs of measure against time or another magni-tude Perce-iving prope-rties of graphs and tables	Building pre-algebraic "geometrical" formula. Choosing an independent variable. Expressing algebraically a domain and a formula	Computing, recognising equivalent expressions. Choosing an appropriate form	Considering 'generic' objects and measures. Interpreting Working on 'families' of functions Parameters (animated or formal). Proving

Table 1

Our grid is sketched in table 1. The organization of rows is based upon the epistemological ground that the mathematical concept of function cannot be separated from the idea of dependency in physical systems where one can observe mutual variations of objects. This necessary connection has also a cognitive foundation: the idea of function is linked to the sensual experience of dependencies in a physical system (Radford 2005). While research generally deals with one or both of these levels, we consider an intermediate level, the level of dependencies between measures or magnitudes that bridges the physical world and the mathematical world of functions. Falcade et al. (2007) for instance focused on the first level: they choose Dynamic Geometry (DG) as a field to provide students a qualitative experience of

covariation and of functional dependency. Arzarello & Robutti (2004) did one of the studies covering the first and third levels, but not the magnitudes. The classroom activities they experimented were about physical movements. Using a motion sensor and a graphic calculator, students produced and interpreted graphs and numeric tables to describe different kinds of motion in term of mathematical functions. The level of magnitudes (distance, time) was actually taken in charge by the calculator: using implicit variables and units for distance and time, it directly transposed the movement into the mathematical world of tables and graphs. Our hypothesis is that activities at this intermediate level can be fruitful for conceptualizing functions: building appropriate variables to quantify observations, distinguishing functional dependencies among more general co-variation, choosing dependant and independent variables… strongly contribute to make functions exist as model of physical dependencies.

The columns in the grid refer to different representations of concepts in calculus adapted from Tall (1996)[1]. We choose to separate representations of relationship between two elements that can be thought of enactively or from images or approximations (enactive-iconic), and those that imply an explicit exact expression and thus an algebraic language. Activities in the columns under the heading 'enactive-iconic' involve experience of movements inside physical systems, as well as work on graphical or tabular representatives of these, and 'explorations' (Yerushalmy 1999) on graphs and tables of approximate values. In these activities, a local point of view is related to what happens 'near a value', and therefore 'local' activities tend to explore physical systems by 'small movements", and tables and curves by tracing or browsing near a point, with adequate zooming in. A global point of view considers properties of dependencies and their representatives on whole intervals. In our meaning, distinguishing the local and global points of view in these activities is essential in the transition to calculus (Maschietto 2008). Research in this column insists upon the complexity of semiotic systems involved in these activities. Falcade et al. (2007) for instance insisted that DG tools (Dragging, Trace, Macro...) and particular signs (segments, rays, figures representing either the domain, on which the independent variable varies, or the range, on which the dependent variable varies) offer a common semiotic system that the students and the teacher can elaborate on.

Activities under the heading 'Algebraic' involve signs and rules specific to Algebra. Student activities in algebra have been classified by Kieran (2004) into three categories: generational, transformational, and global / meta-level that correspond to columns in our table: "*The generational activities of algebra involve the forming of the expressions and equations that are the objects of algebra (...). The transformational (rule-based) activities include, for instance, collecting like terms, factoring, expanding, substituting(…) The global / meta-level mathematical activities*

[1] The columns in our table are the same as Tall's (p. 295) except for the 'formal' representations that we will not discuss here because of lack of space.

include problem solving, modelling, noting structure, studying change, justifying, proving, and predicting."

We are specially interested by how meaning can develop at the interface between the three categories and enactive-iconic activities. Very significant *generational activities* exist at this interface: finding an algebraic expression (domain and formula) for a function is motivated and takes sense when the function is conceived as a model of an enactive phenomenon. We are especially interested by the connections between generational and enactive-iconic activities at the second level (magnitudes): after independent and dependent variables have been built using a formalisation specific to magnitudes, dependencies can be thought of by reasoning on the laws governing the magnitudes in the system and algebraic work can takes place to express this dependency mathematically.

Rich connections also exist between *transformational* and enactive-iconic activities. For instance students can connect the notion of equivalence, central in the transformational activities with the coincidences of graphs. Yerushalmy (1999) designed the VisualMath curriculum, based on specific graphing software, so that students come to understand what operations on equations are legal ones while performing manipulations as a way to conjecture and understand "on screen" results. Transformational activity is a domain where, for many authors, Computer Algebra Systems (CAS) could alleviate problems related to student weaknesses in computational skills. But, as Yerushalmy note, being 'solution tools' they do not support the construction of a visible map of the point of view of the curriculum.

Global-Meta activities include using algebraic means to express generality in the study of physical systems. Modeling a dependency often involves general objects (for instance arbitrary points in a geometric figure) and thus the model is a family of functions. Algebraically, it is expressed by a function whose domain and formulas depends on parameters. Using the model to solve a problem in the physical system brings together algebraic treatment and enactive-iconic interpretation of parameters.

CASYOPEE

Our choice was to build an open computer environment covering comprehensively the activities in the above grid and allowing easy connection between them. Casyopée has two modules: one is a symbolic window and the other a Dynamic Geometry (DG) window (Figure 1). The symbolic window differs from standard CAS, following Yerushalmy's criticism that standard CAS do not support the curriculum. In Casyopée each object has a clear status with regard to the curriculum. While standard CAS' window is a mere memory of commands and feedback, Casyopée's interface displays dynamically the objects relevant for a problem. The symbolic window supports transformational and global-meta activities as well as enactive-iconic activities on functions. An important feature with regard to these activities is that parameters can have two different statuses that a user can switch at any time. One is 'animated': the parameter has a value that the user can change by way of a slide bar.

This status corresponds to the parameter as a placeholder when the student looks at a particular expression or table or graph, or as a changing quantity when (s)he looks at the evolution of forms by operating the slide bar. The other one is ‘formal’: while keeping a value used for graphs and figures as a placeholder, the parameter is treated formally in algebraic transformations.

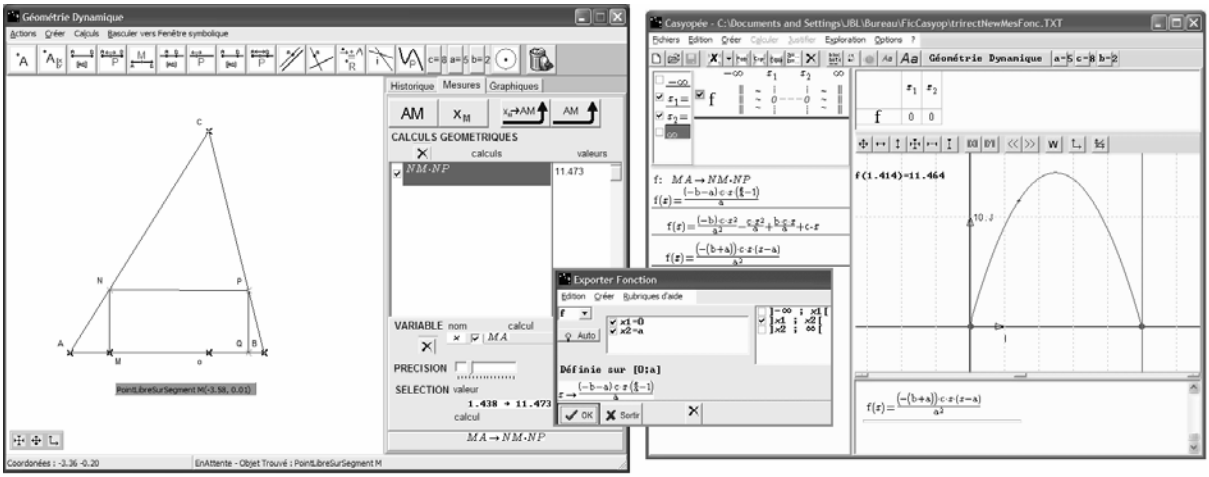

Figure 1: Casyopée’s symbolic and DG windows and the exportation form.

The DG window derives from our choice of geometrical figures as physical systems for enactive activities, consistent with Falcade and al. (2007)’s. It offers the main features of standard DG systems: creation and animation of geometrical objects. It proposes also symbolic facilities for generalisation, parameters entering into the definition of geometrical objects and measures. Finally it offers specific aids for modelling dependencies between magnitudes to respectively create a “geometrical calculation” expressing a measure, select a measure as an independent variable and finally export a dependency between this variable and a calculation, if it exists, as a mathematical function into the symbolic window. These aids ‘reify’ the intermediate level between the figure as a physical system and the mathematical function and allow students to perform a generative activity that would be very difficult without the help of Casyopée.

AN EXPERIMENT

We draw on an experiment in two eleventh grade scientific classes with Casyopée. This experiment is part of the ReMath project. The scenario included first three sessions focusing on capabilities of Casyopée’s symbolic window and on quadratic functions. A second part (two sessions) aimed first to consolidate students’ knowledge on geometrical situations and to introduce them to the geometrical window’s capabilities. Finally, in the third part (one session), students had to take advantage of all features of Casyopée and to activate all their algebraic knowledge for solving an optimization problem. We draw from this last session to show how students developed and connected the activities.

The problem

ABC *a triangle, and o the foot of the altitude from* C *to* AB. *Among the rectangles* MNPQ *with* M *on* [oA], N *on* [AB], P *on* [BC], Q *on* [oC] *is (are) there (a) rectangle(s) with maximum area?*

Figure 1 shows the different features offered by Casyopée to explore and solve this problem. Table 2 shows the main actions that a student can undertake. Casyopée allows going freely forth and back between these actions. For instance a student might explore and model the dependency on a particular figure before using parameters to consider a generic triangle. Although we situated the actions in specific cells related to their main focus, they often involve other representations. For instance the algebraic actions at the level of the figure and magnitudes are deeply linked to the enactive-iconic representations. The aids for modelling dependencies between magnitudes explained above link actions at the level of magnitudes and at the level of mathematical functions: introducing a 'geometrical calculation' for the area (that is to say a formula involving distances, $MN \cdot MQ$ for instance) allows local exploration and conjecturing one maximum, M being the midpoint of [oA]. When a student chooses an independent variable related to M, Casyopée displays possible dependencies between magnitudes in forms like for instance $MA \rightarrow MN \cdot MQ$. The values of the variables are updated when moving M allowing further local exploration and understanding of the formalisation. When the student chooses to export the dependency, Casyopée computes the algebraic domain and formula of the corresponding function. Casyopée offers also means for connecting enactive-iconic exploration at the level of magnitudes and of mathematical functions: a cross on the graph of the function is dynamically linked with the position of point M.

	Enactive-Iconic	Generational	Transformational	GlobalMeta
Geometric figure	Global exploration: various shapes of the rectangle			Considering parameters a, b and c to denote oA, oB and oC
Length and area	Local exploration to conjecture one maximum.	Introducing 'geometrical calculation' for the area	Recognizing a quadratic function, using a method to compute an optimal value.	Working on 'families' of functions
Mathematical Function	Local trace of graph of the function, global recognition of a parabola	Getting an algebraic formula and a domain		Interpreting the generic optimal value

Table 2

Observation

Student's actual paths were very diverse with regard to the time they devoted to each action and the difficulties they had to go from one to another. Some stayed a long time in enactive-iconic exploration. A minority of these students did not really work

on mathematical functions, except for graphs. No student went directly to computing symbolically an optimal value after exporting a mathematical function: they first explored the graph. A few students however went more quickly towards this, some doing no or very little exploration. This diversity of paths is for us an indication that students could work with Casyopee at their own pace, developing activities that they could understand. We now report more precisely on specific students' behavior in connected activities.

Enactive-iconic and generational activities. Many students did mistakes when creating a geometrical calculation for the area of the rectangle. To detect a mistake, they moved M and observed inconsistencies between numerical values or variations of the area and the shape of rectangle. This enactive feedback allowed them to correct the geometrical calculation. The algebraic expression resulting of the exportation of this dependency was another important feedback for the students in the generational activity. Students often changed their choice of an independent variable until they got a sufficiently simple expression. We think that this use of feedbacks, although not always reflective, was an important step in understanding the notion of independent and dependant variable.

Enactive-iconic, transformational and global-meta activities. Students knew a method to compute the maximum's x-coordinate of a quadratic expression, based on the expanded form. They used Casyopée to get this form for the function modelling the dependency. A team got a non-parametric expanded form, because they did not switch the parameters from 'animated' to 'formal'. Then they computed a numerical value for this particular case. While recognizing that they only partially solved the problem, they could not prove the property for a general triangle. Other teams did, but had some difficulty to apply the method to the parametric expanded form. We think that overcoming these difficulties they got an extended understanding of the method, linked to meanings given to the parameters from the geometrical situation.

CONCLUSION

As shown in the paper, the grid was an operational tool inside the ReMath project, for guiding the design of Casyopée and of learning situations. An ambition of ReMath was to progress in connecting and integrating theoretical frames in technology enhanced mathematics learning. Most of the framework involved in this connecting activity such as the theory of didactical situations, the anthropological theory of didactics, activity theory, the theory of semiotic mediation, and the instrumental approach are situated at a rather general level. These general theories were influential but too general for piloting in a precise way the design of artefacts or the use of these. For fulfilling our design needs, we felt the necessity of building local frames, like the grid in this paper. We tried to build coherence among the diversity of approaches to this area, selecting some key perspectives whose complementarities seemed to us potentially productive and organizing these into a structured landscape.

More remains to be done in order to better connect this local level with the global level of macro-theoretical approaches. For that purpose, the system of cross-experimentations which has been developed in ReMath seems encouraging. Casyopée for instance, has been experimented by teams in Italy and in France referring to different macro-theoretical perspectives, respectively the theory of semiotic mediations and the theory of didactic situations. It is thus possible to explore how they combine in didactic action with the local frame we have presented here. We will be able to present more about this at the conference.

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