Connected working spaces: the case of computer programming in mathematics education

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In current approaches of mathematics education at upper secondary level, activities proposed to students involve several fields in interaction. After studying activities about modelling or functions, we question here the development in many mathematical curricula around the world of activities involving computer programming, sometimes labelled "coding" or "algorithmics". The motivation of this paper is that a suitable theoretical framework is required in order to make sense of students' work in activities involving various fields and their coordination, taking into account the semiotic dimension as well as the use of instruments, and the contents and reasoning specific to each field.

Keywords: Connected working spaces, computer programming, fields in interaction, modelling, functions

Introduction

In activities proposed to students involving several fields in interaction, these fields can be related more or less closely to reality or they can be considered as domains of application of mathematics, or they can belong to other scientific disciplines, or they can be mathematical fields like geometry or algebra. With regard to modelling, Lagrange (2018) proposed to consider activities for students involving different models, belonging to different fields with different relationship to reality and to mathematics. Regarding functions, Lagrange and Psycharis (2014) stressed the necessity, in order that students make sense of the idea of functions, of activities in various fields, for instance geometry, measure and algebra. With regard to computer programming, some curricula, especially in France, insist on a potential contribution of the work on algorithms to mathematical understanding, raising again the question of interaction between distinctive fields in students' activity. Few research studies have been done in this area (Lagrange, 2014) and then this paper is based upon the recent doctoral research study carried out by the second author (Laval, 2018). In the next section, we will discuss how current theoretical frameworks analyse various aspects of these activities and their contribution to students' understanding, and we will propose a framework taking into account comprehensively these aspects.

Theoretical developments and question

Classically, activities involving a mathematical task in several fields are analysed by considering that entities involved in the task appear under different semiotic representations, each pertaining to a field. This is the "multi–representation" view. Among the many theoretical approaches of multi–representations, we start from Duval's (2006) consideration of the plurality of representations for a given object. For Duval there is no other ways of gaining access to the mathematical objects but to produce some semiotic representations and he stresses that representations are organized in semiotic systems. In a semiotic system, some representations, called "registers" and there is a need for a specific focus on processes of work inside and between the registers. In this multi–representational approach, activities for students in different fields are considered helpful because of the opportunities they offer for working on different semiotic representations and coordinating these.

In spite of the usefulness of frameworks like Duval's, the "multi–representation" view is for us too reductively semiotic and cannot alone make really sense of activities involving several fields in interaction, and of their potentialities. In some curricula, much emphasis has been put on the work on representations and students can be fluent in the processes of conversion and treatments, but this does not necessarily imply a deep understanding of notions at stake. For instance, even when students are proficient in dealing with the four classical representations of functions (verbal, symbolic, graphic and tabular), fundamental aspects of functions (correspondence, co–variation, mapping…) and their coordination remain problematic (Ayalon,Watson & Lerman, 2014).

Another framework for making sense of activities of coordinating different fields (especially mathematical fields) is by Douady (1986). For Douady, a setting is constituted of objects from a branch of mathematics, of relationship between these objects, their various expressions and the mental images associated with these objects. When students solve a problem, they can consider this problem in different settings. Switching from one setting to another is important in order that students progress and that their conceptions evolve. According to authors like Perrin–Glorian (2004), it is sometimes difficult to distinguish the representational and the settings approaches, especially when a phase of work can be thought of both as a switch between settings and as a conversion of representations. Actually, rather than contradicting, the two approaches complement: beyond its mathematical contents, each setting offers specific semiotic systems, and coordinating the settings also implies coordinating the semiotic systems. For us, this framework is potentially productive in the sense that, beyond representations, it puts the emphasis on mathematical contents and reasoning, and to the coordination of these across different branches of mathematics. However, many of the fields involved in students' activities are not mathematical fields. Taking again the example of functions, this notion is present in several branches of mathematics with different definitions and properties, but these definitions and properties have a meaning only when they are connected with practices in non–mathematical domains: everyday bodily experience, physical mechanisms, etc.

Another concern is how instruments are taken into account in the students' mathematical activity. Twenty years ago, sophisticated calculators became available for students' work, and a framework was developed: the instrumental approach of the use of digital technologies to teach and learn mathematics. This approach has been inspired by research work in cognitive ergonomics but researchers like Lagrange (1999), Artigue (2002), insisted on the intertwined development of knowledge related to the instrument and of knowledge about mathematics in an instrumental genesis. This is important because otherwise an instrumental approach would be only a psychological framework with little insight for mathematics education. Authors like Bartolini Bussi & Mariotti (2008) also noted that the use of instruments and the associated reflection involve a lot of signs that, for a student, may have not immediately a mathematical meaning, and they propose the idea of "semiotic mediation" to refer to the classroom activity necessary in order to ensure the productivity of the work with instruments at a semiotic level.

Each framework, multi–representation, coordination of mathematical settings and instrumental approach, puts a specific focus on the semiotic processes or on the contents and reasoning or on the use of instruments. What would be a framework taking into account comprehensively these three dimensions and able to address work in mathematical fields as well as in non–mathematical fields where mathematical notions can take sense? In this paper, we present the framework of connected working spaces, proposed by Minh & Lagrange (2016) and by Lagrange (2018) for activities related respectively to functions and to modelling and we question its utility for addressing the new challenge brought about by activities involving computer programming in mathematics education.

Connected working spaces

The framework of the Mathematical Working Spaces (MWS) allows characterizing the way the concepts make sense in a given work context. According to Kuzniak & Richard (2013) a MWS is an abstract space organized to ensure the mathematical work in an educational setting. Work in a MWS is organized around three dimensions:

- Semiotic: use of symbols, graphics, concrete objects understood as signs,
- Instrumental: construction using artefacts (geometric figure, graphs, program..)
- Discursive: justification and proof using a theoretical frame of reference.

Activities considered in this paper involve several fields, and for each of these fields, a working space. The framework of "connected working spaces" has been introduced in order to give account of how connections between MWS bring meaning to the concepts involved. This extended MWS framework takes into account the semiotic and instrumental dimensions as well as the contents and mode of reasoning, in different fields of activity and their interaction in a mathematical activity. Then it is not contradictory with the theoretical developments outlined above, but it rather aims to organize them in a comprehensive structure. Researchers who adopt the MWS framework emphasize circulation (or 'zigzagging') between the "vertical planes" (i.e. the two by two combinations of semiotic, instrumental and discursive dimensions) as means to give account of how semiotic, instrumental and validation processes are conducted by students in a comprehensive and coherent way. In the framework of "connected working spaces", we pay special attention to how connections between working spaces involve these combinations of dimensions (Lagrange, 2018).

Connecting Algorithmic and Mathematical Working Spaces: The Intermediate Value Theorem (IVT)

Activities involving computer programming in mathematics education involve two distinct working spaces an Algorithmic Working Space (AWS) and a Mathematical Working Space (MWS). In the continuation of this paper we will focus on a particular topic that can be considered both from a computer science and a mathematical points of view: the solution of an equation $f(x)=0$ for a given function f defined on a closed interval [a, b]. We will consider this topic relatively to how it can be a subject for secondary students' work, that is to say how it implies connecting the two working spaces.

From a computer programming point of view, specific algorithms allow approaching solutions as close as possible. We consider algorithms able to find, for an arbitrary precision e, an interval $[u;v]$ with the property P(e): $|u-v|<e$ and $f(u)\cdot f(v)\leq 0$. The simplest algorithm scans iteratively the sub intervals of length e, until finding a suitable one (Figure 1 left). A more efficient algorithm is

based on dichotomy (Figure 1 right). In the corresponding AWS, the semiotic dimension is characterized by specific marker of iterative (While…) and alternative (If…) treatments and by variables, whose value, as a difference with mathematical variables, change along the treatment by way of the specific operation of affectation (sign ← in Figure 1). Mathematical expressions are also involved. The associated semiotic system can be at stake for students not fluent in algebra, especially the notation *f*(…). There is a strong instrumental dimension, since algorithms are intended to be executed by an automatic device and it is expected that execution will help students to make sense of the formalism. The discursive dimension is characterized by questions like the termination of an algorithm (does it terminate in a finite number of steps?), its effectivity (does it return an appropriate solution?) and its efficiency (how many steps are necessary for a given set of data?) More or less formal reasoning allows to prove that, with the condition $f(a)\cdot f(b)<0$, both algorithms terminate and return an interval with the property P(e), and to compute the maximum number of steps for the scan algorithm, and the exact number of steps for the dichotomy algorithm.

A scan algorithm	A dichotomy algorithm

$u \leftarrow \mathrm{a}$
$v \leftarrow \mathrm{a}+\mathrm{e}$
While $f(u)\cdot f(v)>0$
$u \leftarrow u+\mathrm{e}$
$v \leftarrow \min(b, v+\mathrm{e})$
End While
Return $[u;v]$
	$u \leftarrow \mathrm{a}$
	$v \leftarrow \mathrm{b}$
	While $v-u>\mathrm{e}$
	$\mathrm{m} \leftarrow \frac{u+v}{2}$
	If $f(u)\cdot f(m)>0$ then
	$u \leftarrow \mathrm{m}$
	Else
	$v \leftarrow \mathrm{m}$
	Endif
	End While
	Return $[u;v]$

Figure 1: algorithms for approaching a solution

From a mathematical point of view, the Intermediate Value Theorem (IVT) guarantees the existence of a solution on the interval [a;b] under the sufficient conditions: *f* is continuous and $f(a)\cdot f(b)<0$. A corollary ensures that, under the sufficient condition that *f* is monotonic, the solution is unique. The corresponding MWS has a strong discursive dimension: it includes properties of functions like continuity and monotonicity; it is focused on a mathematical solution, rather than on a process of approximation. A classical proof is based on two adjacent sequences. In addition to the usual mathematical formalism, the semiotic dimension is then characterized by the formalism of infinite sequences, different from the iterative variables of the algorithms, although both are defined by way of the dichotomy method. Students are introduced progressively to these notions and formalism from 10^{th} to 12^{th} grade. The instruments here are paper and pencil calculations, and graphical display of functions.

Organizing the working spaces: a classroom experiment

The outcome of the above analysis is that the algorithms and the theorem have different targets: while the IVT is about solutions, the algorithms aims at obtaining an interval with the property P(e). However, there are clear links. First, the IVT ensures that, with the sufficient condition of continuity, an intervals with the property P(e) actually contains one or more solutions, and the corollary that, with the additional sufficient condition of monotonicity, it contains the unique solution. Second, the algorithms, especially the dichotomy algorithm provides a mode of generation of sequences that play a crucial role in a proof of the IVT. From these links different organisations of the MWS and the AWS can be envisioned. In a first organization, after being taught about the IVT, students can work on the algorithms with a function verifying the sufficient conditions in order to get an approximation of the solution. We name this organisation “application”: computer programming is considered as an application of “pure” mathematics. This is the most common scheme that we found when looking at textbooks in France. As for us, we envision other organisations making the working spaces interact more closely. It is because, as we wrote above, knowledge is at stake in both working spaces, and interaction can help understanding. Working on an algorithm, and after proving that the returned interval has the property P(e), students can experiment on diverse functions, in order to infer sufficient conditions for the IVT. They can then work on a proof of the theorem by conceiving adjacent sequences from the iterative variables in the algorithm.

Classroom situations

We designed classroom situations in order to test the hypothesis that, transitioning from “application” to other organisations, students make connections between working spaces in the various dimensions. The situations were implemented in three French classes at 10th, 11th and 12th grade in order to get evidence about the work of students with different mathematical attainments. Each class had around 30 students and had nothing particular with regard to the work expected. The duration of each situation was between ½ and one hour. The students had worked before on the dichotomy method for discrete numbers (Laval, 2016) and this work was mainly in the AWS. Otherwise they had no previous experience in the domain, except that, for the 12th graders, the IVT had been introduced and not proved.

The first situation was “application”: a continuous monotonic function was given and the dichotomy method was exposed by way of a flow chart. The students had to give some evidence of the existence of a unique solution, and to implement the method for this function in a textual programming environment allowing execution. The situation was intended to make students work in the AWS and MWS and coordinate these especially in the semiotic and discursive dimensions.

In the second situation, with the same function and the same programming environment, students had to complete a scan algorithm where the condition of continuation (following While) was missing. While this situation clearly involves the AWS in the three dimensions, the MWS is in the background, both with regard to the formalism and to the properties of the function.

The third situation dealt with sufficient conditions for existence and unicity of a solution. The goal was to make students aware of functions for which the dichotomy algorithm does not return a suitable interval. They were requested to implement and execute a dichotomy algorithm for two functions, and to answer the question “does the interval returned by the algorithm actually contain the unique solution?” The expressions of the functions had been entered in the programming environment but were hidden Thus the students could refer to the functions only by the notation *f*(…). One function had been chosen continuous but not monotonic, and the equation had two solutions over the interval. The other function had a pole inside the interval (i.e. it was not defined for a value, and has infinite left and right limits). The task is reflective, both in the AWS and the MWS: evaluating the effectivity of the algorithm at a mathematical level. The students were expected to be influenced toward an affirmative answer by the “application” situation where mathematical effectivity was not discussed. Students at 12th grade knew the IVT, but were expected not to focus on sufficient conditions, because all examples treated before were continuous monotonic functions. However, the students were expected to double check by graphing the functions, or computing values. This connection between the AWS and the MWS involves the instrumental and the discursive dimensions.

The fourth situation was implemented only for 12th grade students. Students were invited to build a proof of the IVT, using adjacent sequences and the dichotomy method. The semiotic dimensions of the MWS and the AWS are at stake in this task, with a process of conversion, from the iteration on variables in the AWS, to sequences in the MWS. In the discursive dimensions the convergence of the sequences had to be inferred from the fact that P(e) holds for arbitrary e. However, the convergence does not prove that the limit is a solution, and students were expected to use explicitly a theorem on continuous functions and sequences, and another about the compatibility of limit and order. This work is specific to the MWS, but had been prepared by the focus on sufficient conditions in the third situation.

Observation and evaluation

In the first situation, 10th graders gave graphical evidence for the existence of a unique solution, while 11th and 12th graders performed an analytic proof of monotonicity. This is consistent with methods taught at these levels. 12th graders mentioned that the function changes it signs over the interval, but not the continuity, although the IVT had been taught to them. The implementation of the method raised a lot of difficulties in the semiotic dimension of the AWS. The status of the function in the algorithm was also difficult. Many students did not know how to use the notation *f*(..) and tried to enter the explicit expression, but they were confused, thinking that the result should be a string rather than a number. The conversion of structure from flow chart to text implies changing the expression of the condition in the iteration, and students often relied on the execution to find a suitable expression. Then the instrumental dimension in the AWS was a scaffold to semiotic difficulties.

In the second situation, most students used the fact that the function was increasing (it was the function of the first situation), to obtain a simplified expression of the condition of continuation in the scan algorithm ($f(v)<0$, rather than $f(u).f(v)>0$). This is evidence that the students reflected on the actual function rather than on a general algorithm. At 10th grade we observed again syntaxic difficulties linked to the semiotic of assignment and functions in the programming environment.

In the third situation 10th and 11th graders considered that the algorithm with the first function returned an interval actually containing “the” solution. 12th graders were dubious because the condition that the function changes it signs over the interval was not satisfied. After graphing the function, most students were very surprised that the equation had two solutions. 10th graders generally said that the algorithm was wrong “because there should be only one solution”. It means that, for them, the termination of the algorithm should be an evidence of a unique solution. 11th and 12th graders rather said that the algorithm should return two intervals. In both cases, students over interpreted the possibilities of an algorithm with regard to a mathematical goal. Few students, mainly 12th graders, looked at the execution of the algorithm and observed the intervals obtained through the iteration on the graph in order to explain why the interval returned after the execution includes one particular solution rather than two. Work in the instrumental dimension (execution of the algorithm and graphing) clearly helped them to operate the delicate coordination of the discursive dimensions in the AWS and the MWS. With the second function, the students again considered that the very small interval returned by the algorithm was an evidence of the existence of a single solution. Most 10th graders reconsidered this finding after graphing and considering the unusual shape of the graph. However, some misinterpreted the vertical pseudo–asymptote at the pole, an artefact of the graphing algorithm, as a part of the graph and concluded at a solution at the intersection with x–axis. 11th and 12th graders were more suspicious and calculated the values of the function at the boundaries of the intervals returned by the algorithm for decreasing values of the threshold e. They found values of the function increasing at the left boundary and decreasing at the left boundary. They deduced that these intervals approach a pole rather than a solution. The outcome of this third situation is that, except for a few 10th graders, the students made a clear distinction between the effectivity of the algorithm in the AWS and its effectivity to approach a solution in the MWS. 11th and 12th graders had a notion of sequences and convergence that helped them to consider more closely the phenomenon.

As mentioned before, the fourth situation was implemented only at 12th grade. At the beginning, the students were confused, not connecting sequences and the IVT which they thought related to functions. Then some of them proposed to look at the values of the boundaries of the intervals along the execution of the dichotomy algorithm for a particular function. This is a typical answer:

> The sequences (un) and (vn) are adjacent because (un) is increasing, (vn) is decreasing and (vn–un) becomes closer to zero when n becomes bigger and bigger. Then these two sequences converge towards a common limit c. Because *f* is continuous, *f*(un) and *f*(vn) converge towards *f*(c) which is zero. The theorem is proved by way of the computer for a particular function.

This “proof” is a mix of observation (behaviour of the sequences, value of *f*(c)) and deduction (convergence of the sequences) and, for the students it is valid only for one function. Within the duration of this situation students could not go much beyond. Only one observed that a proof of the behaviour of the sequences could be made by induction. For us, the students adequately took advantage of the work in the AWS but in some way stayed halfway between the AWS that produces evidence on an example, and the MWS in which a formal proof for a generic function was expected. Students had no difficulty to operate the semiotic conversion from computer variables to sequences. In contrast, their answer witnesses a notion of proof still confusing instrumental evidence and mathematical reasoning.

Conclusion

This paper questions the usefulness of the connected working spaces framework for addressing activities for students involving distinctive fields and especially computer programming and mathematics. We used this framework for designing a classroom experiment to test a hypothesis: for a particular topic involving computer programming and mathematics, it is possible to characterize an AWS and a MWS, and to create situations in order that students make fruitful connections between these working spaces in the three dimensions, semiotic, instrumental and discursive. We observed a variety of connections validating this hypothesis. In the first and second situation, work in the instrumental dimension was a scaffold to semiotic difficulties at the boundary of the AWS and the MWS. In the third situation it also helped students to operate the delicate coordination of the discursive dimensions in the AWS and the MWS. In the fourth situation the students took advantage of the work in the instrumental and discursive dimensions of the AWS for their discursive work in the MWS, although they were only partially successful. Previous studies about functions and modelling already gave insight into a potential of the connected working spaces framework and this paper extends the analysis to computer programming and mathematics education. It also witnesses of a framework that do not contradict with other approaches like multi–representation, settings and instruments but rather connects these in a comprehensive analysis of students' work.

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