A framework for analysing students' learning of function at upper secondary level: Connected Working Spaces and Abstraction in Context

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Abstract. In this chapter, we take a networking perspective to develop a framework for analysing classroom situations and students' evolving conceptualisation of function as covariation by combining Connected Working Spaces (CWS) and Abstraction in Context (AiC). We privilege authentic modelling tasks and the use of digital environments offering integrated algebraic and geometrical representations of functions to address students' evolutions in the path from physical context to algebra. The combined analyses based on the two frameworks allow a deeper look at students' cognitive evolution as they experience functions in a plurality of settings (physical context, geometry, measures, algebra).

Keywords: Functions and digital technology, Networking of theoretical frameworks, Connected functional working spaces, Abstraction in Context, Authentic modelling.

INTRODUCTION

This chapter is about a new framework to make sense of learning situations concerning functions at upper secondary level created by networking two theoretical approaches. The authors are researchers who share a common interest in students' approaches to functions, however, they have different research interests and theoretical backgrounds. While the third author is interested in the design and evaluation of classroom situations with broad assumptions regarding students' learning, the others pay more attention to students' evolving conceptions through their activity along the situations. Our approach to networking stems from the suggestion found in Artigue et al. (2006) that "as a research community, we need to be aware that discussion between researchers from different research communities is insufficient to achieve networking. Collaboration between teams using different theories with different underlying assumptions is called for" (p. 1242).

Two authors have collaborated previoulsy (Lagrange & Psycharis, 2014) and implemented a "double analysis" method to analyse studies based on the use of digital environments that offer integrated geometrical and algebraic representations. This double analysis resulted in a deepened and more balanced understanding about the nature of learning situations for functions and the process of function conceptualisation by students. Meanwhile the two authors continued their theoretical work with each adopting new theoretical perspectives, *Abstraction in Context (AiC)* (Dreyfus, Hershkowitz & Schwarz, 2015) on the one hand and *Connected Working Spaces (CWS)* (Minh & Lagrange, 2016) on the other. The work presented here goes beyond the double analysis because the first two authors collaborated with author three to design, implement and analyse a series of classroom situations as a basis for the latter's doctoral work. The whole experiment was carried on articulating the two frameworks. At a practical level, authors one and two conducted the experiment in their school context and adopted AiC. Furthermore, they made use of Casyopée (Minh & Lagrange, 2016), a digital environment that was designed and widely experimented in the context of author three adopting CWS. Beyond this experiment, our collaboration aimed to develop a framework for analysing classroom situations and students' evolving conceptualisation of functions. We privilege tasks involving modelling situations and the use of digital environments that offer integrated algebraic and geometrical representations of functions. Our focus is on students' transition from experiencing dependencies in non-algebraic (digital and non-digital) settings to expressing and working on these dependencies mathematically.

APPROACHES TO FUNCTIONS AT UPPER SECONDARY LEVEL

The domain of functions occupies a central position in school mathematics curricula, however reaching a suitable understanding of functions is rather difficult for many students. Research studies have evidenced students' difficulties in understanding function as covariation and in linking work on magnitudes (or "quantities' measures") with mathematical functions (Thompson & Carlson, 2017). Existing research has also indicated that at both epistemological and cognitive levels functions take sense through experiences in many dissimilar settings. However, teaching approaches to functions at upper secondary level overemphasise the use of "algebraic" representations (i.e. formulas, graphs and tables) and senseless manipulation while overlooking situations, tools and resources that could facilitate students' meaningful engagement in functional thinking (Minh & Lagrange, 2016).

At the level of curriculum and textbooks, functions are often approached through tasks involving extra-mathematical situations (e.g., real-world problems) and use of digital tools. A closer consideration, though, reveals that these tasks are mainly applications of already taught knowledge favouring the use of classical "algebraic" methods (Robert & Vandebrouck, 2014). In contrast according to Kaiser and Schwarz (2010), authentic modelling situations are characterised by problems that are only slightly simplified and are recognised by people working in a field as being problems they might meet in their daily activity. Also with regard to modelling and functions, Lagrange (2018) proposed activities where students engage in working with models that have different relationships to reality and to mathematics, paying particular attention to how to help students make connections between these models. In this paper, we adopt a similar view that favours using modelling tasks based on authentic situations as a context for studying students' functional thinking and pays specific attention to the different settings involved in the complex path from physical context to algebra. In this path, a function exists first as a dependency between physical objects, then between geometrical objects, then between quantities and finally, as a mathematical function.

THEORETICAL FRAMEWORK

Connected Working Spaces

Classically, activities involving mathematics and other settings are analysed by considering that entities involved in the task appear under different semiotic representations, with each one pertaining to a field. In this multi–representational approach (e.g., Duval, 2006), activities for students in different settings are considered helpful because of the opportunities they offer for working on different semiotic representations (or *registers*) and coordinating these. In spite of the usefulness of frameworks like Duval's, for us the "multi–representation" view is too reductively semiotic. Standing alone, this view cannot really make sense of activities that involve several fields in interaction and of their potentialities. Douady (1986) offered another framework for making sense of activities of coordinating different settings (especially mathematical ones). For Douady, a setting is constituted of objects from a branch of mathematics; a relationship between these objects and their various expressions; and the mental images associated with these objects. When students solve a problem, they can consider it in different settings. Switching from one setting to another is important for the evolution of students' conceptions. For us, this framework is potentially productive in the sense that beyond representations, it puts the emphasis on mathematical contents and reasoning and their coordination across different settings. Another concern is how instruments are taken into account in students' mathematical activity. Twenty years ago, sophisticated calculators became available for students' work and a framework was developed: the instrumental approach of the use of digital technologies to teach and learn mathematics. This approach has been inspired by research work in cognitive ergonomics, but researchers like Lagrange (1999) insisted on the intertwined development of knowledge related to the instrument and of knowledge about mathematics in an instrumental genesis.

Each framework, multi–representation, coordination of mathematical settings and instrumental approach focuses on a specific dimension: the semiotic processes, the contents and reasoning, and the use of instruments. We retain the idea of *"Mathematical Working Spaces" (MWS)* because it offers a framework associating the three dimensions. According to Kuzniak and Richard (2014) a MWS is an abstract space organised to ensure the mathematical work in an educational setting. Work in a MWS is organised around three dimensions: semiotic (symbol use, graphics, concrete objects understood as signs); instrumental (construction using artefacts, such as geometric figure, graphs etc.) and discursive (justification and proof using a theoretical frame of reference). CWS builds on this idea of MWS by considering that in activities involving mathematics and other settings, students have to work in several working spaces and to coordinate the semiotic, instrumental and discursive dimensions of these spaces. This applies especially in the approach to functions and in modelling.

Abstraction in Context

Building on Freudenthal’s idea of “vertical mathematisation”, AiC defines abstraction as a process of vertical reorganisation of some of the students’ previous mathematical constructs within mathematics and by mathematical means in order to lead to a new construct for the learner (Dreyfus et al., 2015). In this process, previous constructs serve as building blocks for building new constructs, thus students’ construction of mathematical meanings resides in the verticality of the knowledge construction process and the depth of the resulting constructs. Sequences of problem situations can provide fruitful contexts to explore if and how new constructs operate as potential building blocks for further constructions.

AiC describes the process of abstraction by means of a model of three epistemic actions that researchers can be observe and analyse: recognising (R), building-with (B) and constructing (C). Recognising an already known mathematical concept, process or idea occurs when a student recognises it as inherent in a given mathematical situation and relevant to the problem situation they are dealing with. Building-with comprises the combination of existing knowledge elements (i.e. recognised constructs) to achieve a goal, such as solving a problem or justifying a solution. Constructing is carried out by integrating previous knowledge elements (constructions) by vertical mathematisation to produce a new structure/construct (RBC-model, Dreyfus et al., 2015). The model suggests constructing as the central epistemic action of mathematical abstraction and indicates both the process of abstraction and its outcome expressed or used by the student for the first time. Students become aware of their constructs and are more flexible in using them through consolidation, a process that is likely to occur when students build with a previously emerging construct in subsequent activities. In the process of abstraction, the epistemic actions are nested dynamically, that is, R and B actions are nested within C-actions while the R-actions are nested within B-actions. Although this process describes an essentially cognitive approach, AiC attributes special attention to the role of contextual factors (e.g., resources, modelling) in learning. In studies involving authentic modelling and use of digital tools like the present one, the analysis based on RBC actions needs to take into account the richness of the available resources.

Combining AiC and CWS

Aiming to explore the different settings in which functions can make sense, Minh and Lagrange (2016) considered three CWS and analysed students’ and teachers’ evolution concerning functions in terms of how they understand the affordances provided by each of these spaces. Theses researchers’ analysis highlighted a gradual construction of personal spaces and the connections between them and implied cognitive evolution of the subjects. However, in the CWS framework, precisely identifying subjects’ cognitive evolution remains an open issue. It is then expected that identifying AiC’s epistemic actions in episodes indicating the progressive construction of personal spaces will help to make sense of the subjects' cognitive evolution. Then CWS can be seen as providing a "horizontal view" of students' progress along a path from physical settings to mathematical functions, while AiC provides a "vertical reorganisation view" of knowledge construction at key stages of the path. The two frameworks give some importance to "a priori analysis" in order to orient the enactment of an experimental learning situation and as a basis to make sense of the observations. Consistent with the above horizontal view, we expect from CWS an a priori analysis that precisely identifies working spaces and the three dimensions (semiotic, instrumental, discursive) in each one of these and helps to foresee students' and teacher's work and the possibility of productive connections. Consistent with the above vertical view, we expect from AiC an a priori analysis identifying knowledge elements, "typically concepts or strategies thought of in terms of the content domain" (Dreyfus et al., 2015, p. 191). Our research questions are:

(1) How can AiC and CWS help to appreciate the potential of an implementation of an authentic modelling problem?

(2) How can combining AiC and CWS help to make sense of students' conceptualisation of function as covariation in different settings while working on an authentic modelling problem?

METHOD

The context

The research reported in this paper is the first part of an ongoing classroom-based design research that aims to study meaning generation for function as covariation by 16-year-old students working in groups with concrete materials (e.g., manipulatives) and software environments to model a series of real life situations. The experiment we present here took place in a secondary school with one class of twenty 11th grade students (10 groups of two or three). One researcher acted as teacher (called teacher in the paper) and another had the role of participant observer in the classroom. The class had 14 teaching sessions (45 minutes each) over 3 months (one teaching session per week). At the time of the study, the students had been taught about function as correspondence (according to the curriculum), monotonicity and extreme points.

The choice of a problem as a basis for a situation

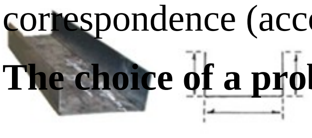

Fig. 1 Gutter design

The gutter problem. People designing gutters made of a sheet of metal to channel rain water from a roof aim to fold the sheet in order to maximise water flow. Such designers know that maximising water flow is achieved by maximising the cross-sectional area and that the sides of the rectangle (L and l) are linked by the relationship L+ 2l= C (C being a side of the sheet of metal). Then, they use mathematics to establish that an optimal shape is for L=2l and l= C/4, L=C/2.

For secondary students, the main idea at stake is covariation emerging from authentic constraints and the conception evolves through the consideration of successive models. In considering these models, specific attention is given to the problematic nature of coupling reality and mathematics. An essential initial step of modelling concerns developing a clear understanding of the reality that is to be represented mathematically. Thus, a **first model** stays close to reality: A rectangular sheet of paper can be folded in order to make a rectangular cross-section and one can appreciate sensually how the choice of the folds influences water circulation. The covariation in this model is between concrete entities (i.e. folds, flux), but the variations are difficult to appreciate because of the poor dynamicity of the model. **A geometric model** derives from the recognition that the rectangular cross-section is what has to be optimised. Dynamic geometry (DG) adds dynamicity and interactivity and helps students become aware of variations. A single free point has to be created that allows constructing the other "dependent" points of the rectangle. Constraints have to be set on this point and on the other points in order to reflect the real cross-section's constraints. This DG model involves quantities in the expression of the constraints. However, covariation is between geometric entities, points and rectangle. Thus, while moving the free point, one can observe the variations of the rectangle: the rectangle is "flat" for the two limits of the variations of the free point, while it grows and decreases as the point moves from one limit to the other. The next important step is a **model involving quantification**. Thompson and Carlson (2017) consider this as a rather complex passage for students. After overcoming difficulties, though, observation of quantities in this model (rectangle area, side length) should allow students a better understanding of covariation, meaning: covariation between quantities rather than between geometric entities. In a **final model**, a selected relationship between quantities (i.e. a length of the rectangle, area) can be expressed by the mathematical formalism of functions. This is again a non-obvious step as it requires understanding the notions of independent and dependent variables and working with algebraic registers. This model allows applying classical techniques for solving the problem.

The choice of a digital environment

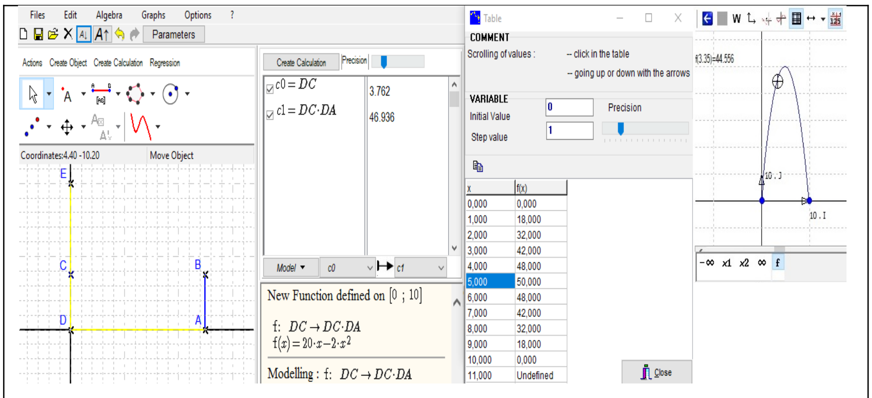

Fig. 2 DG, geometric calculations, table and graph

The Casyopée software (Fig. 2) has been chosen because it offers both a DG window and a symbolic window with registers: tabular, graphic and symbolic allowing students to work on a geometric model as well as on the final algebraic model with the same environment.

Moreover Casyopée offers a “geometric calculations” tab: students can define independent and dependent quantities as measures in a “geometric calculations” tab and investigate their covariation. Minh and Lagrange (2016) explain the various relationships that can be made between the DG window and the symbolic window as well as with the “geometric calculations” tab, consistent with the idea of CWS. Especially, a pair of measures in functional dependency can be exported to the symbolic window as a function that can be further treated by the students using formulas, tables and graphs. This functionality –called “automatic modelling”– is designed to help students in modelling dependencies.

The activity sequence

The activity sequence started by an introductive phase (2 teaching sessions): after an introduction to the main features of Casyopée (e.g., dependencies by moving free points, definition of geometric calculations, automatic modelling), the students were presented the problem of the closest point on a parabola from a given point (Minh & Lagrange, 2016). Then, in the next 12 teaching sessions the students were engaged in modelling three realistic situations through the tasks: *Gutter*, *Front of a Store*, *Oil Tank*. In this paper, we analyse the first task and its implementation (4 teaching sessions).

The task presented to the students

In the *Gutter* task, the students were engaged in exploring the aforementioned gutter models. The task was divided in four steps corresponding to students’ activities in each model: (1) experiment with folding a piece of paper (10cmX20cm) to explore and notice dependencies; (2) construct a dynamic DG figure that models the situation in Casyopée and explore; (3) use the software to propose quantities in functional dependency; (4) obtain a mathematical function modelling the dependency and use algebraic techniques. The task can be considered as a case of authentic modelling since the problem directly makes sense in a workplace field and replacing metal by paper does not oversimplify the problem. Moreover, the task is designed so as students could make sense of the situation and develop an understanding of how a range of tools (including Casyopée) can be used to provide different models.

A PRIORI ANALYSIS OF THE SITUATION

CWS analysis

We characterise four working spaces and their connections in order to foresee what students’ work can be in a classroom situation based on the gutter problem.

The “sheet of paper” working space. There is a strong instrumental dimension: folding helps to approach a solution and to conceptualise the covariation. The discursive dimension involves argumentation about the rectangular cross-section as representative of the flux. In the semiotic dimension, it is expected that the vocabulary will be consistent with the “real life” situation: bottom, edge, water flow, cross-section, etc. These three dimensions of the work are linked to the realistic context, which is for us a characteristic of authentic modelling.

The “dynamic geometry” working space. There is also a strong instrumental dimension in this working space created through the use of a DG environment. Building a model in this dynamic geometry working space is far from obvious for students, since the idea of dependency at stake in this situation underpins the construction. The semiotic dimension is in the geometrical objects (free and dependent points, segments…) used to denote important elements of the gutter. As a difference with the sheet of paper, the constraints have to be expressed explicitly, using the capabilities of the DG window, and this implies adopting the associated semiotics.

Connection with the sheet of paper working space. The experience of the previous model of the sheet of paper should help students to understand and express the involved constraints. In this working space, variations are more easily observed and this should help argumentation in the discursive dimension.

The “measure” working space. We consider the work on quantities in a specific working space due to the gap existing in students’ conceptions between covariation of geometrical objects and covariation of quantities. As explained above, in Casyopée quantities are handled in a “geometric calculation tab”. This allows forming and numerically exploring “pre-algebraic” formulas that represent geometrical quantities. For instance, in the gutter problem, the area of the rectangle has to be expressed as the product of two lengths. Then the semiotic dimension is characterised by this “pre-algebraic” symbolism. This tab also offers the possibility of selecting two formulas (i.e. one for the independent variable and the other for the dependent variable) and checking whether the corresponding quantities are in functional dependency. Then the argumentation in the discursive dimension can go beyond covariation between geometrical objects towards dependency between quantities. The principles at stake in the argumentation deal with dependencies relative to a dynamic figure that are at the core of the idea of function: A dependency has to be expressed by way of quantities that are measurable attributes of a geometrical object and one quantity (the independent variable) has to depend univocally on this object.

Connection with the dynamic geometry working space. The measure working space introduces a new pre-algebraic symbolism and new means to work on a dependency. The action (drawing a point, observing variations, etc), though, is still close to the work in DG. This should help students to grasp how quantities with their specific symbolism are related to geometrical objects and how quantifying helps to make sense of a dependency.

The “mathematical functions” working space. This space is consistent with the “traditional” paper-and-pencil school culture involving analytic functions, algebraic notation, graphs, tables and transformations conserving equivalence. However in this culture, the idea of function is often overlooked because of an overemphasis on algebraic manipulation. We expect that in this working space students will reflect on representations in three registers (formulas, tables, graphs), rather than on the application of algebraic techniques. Casyopée's capabilities contribute to the instrumental dimension of the work in this space. The discursive dimension is again about the variations. However, it differs from the other spaces in that it adopts the standard mathematical way of reasoning (independent/dependent variables, maximum value) and the semiotic dimension is also the standard mathematical registers.

Connection with the measure working space. We expect that students will associate the ideas of mathematical function and of dependency between measures at the level of formalism (for instance associating a formula like $x(l-x)$ with the product of two lengths *(DC x DA)* and at the level of the understanding of variations associating the dependent and independent variables with quantities linked in a particular way.

AiC analysis

The implementation in four steps intends to make covariation appear in the four different models and the corresponding working spaces (presented above). In terms of AiC, the knowledge elements are related to these working spaces.

First step. Students should have to conceptualise the important elements of an authentic situation (i.e. ways of folding, volume and cross section) by recognising the opportunities offered by the sheet of paper working space, experimenting with the model by folding it at their own initiative and constructing an understanding of the variations (i.e. relating the gutter sides to the problem, recognising the cross-sectional area as suitable for maximisation). At each one of the next steps, the students are expected to recognise means offered in a new working space as corresponding to constructs of the previous model in order to build a new model and construct new knowledge elements relatively to covariation and functions. This recognition is expected to be feasible, thanks to the connections between the working spaces identified above.

Second step: In order to use the DG environment, students should have to identify geometrical objects corresponding to elements of the paper sheet model and organise these objects in order to build a DG model consistent with the paper one. The corresponding knowledge elements are: (2a) the idea of independent geometrical entity necessary to build a dynamic model and (2b) the idea of covariation between geometrical entities. We shall say that the students have constructed these elements, if they identify the need to use one side of the model as an independent entity, specify the corresponding restrictions for the points and express their coordinates accordingly using the available symbolic notation.

Third step: Students should first recognise quantities at stake in the DG model (area of the rectangle, side length) and then covariation and dependency between these quantities. The knowledge elements here are: (3a) covariation of quantities; and (3b) formalisation of this covariation as a pair of variables. We shall say that the students have constructed these elements, if they work with pairs of covarying quantities in the automatic modelling tab and express their understanding of the role of dependent and independent variables and their covariation.

Fourth step: Students should conceptualise mathematical variables representing the quantities of the preceding model and a mathematical function representing the covariation of these quantities. The knowledge elements here are: (4a) the idea of a function modelling a covarying pair of variables and (4b) mathematical techniques of working with this function using its standard registers (graphic, tabular, formal). We shall say that the students have constructed these elements, if they create a function from a covarying pair of variables and explore the problem further using relevant mathematical registers.

DATA AND ANALYSIS

The collected data consists of video and audio recordings (four groups) that were fully transcribed. In the first level of the analysis, we selected episodes in which the students referred to covarying magnitudes for every different working space. In the second level, the selected episodes were analysed twice according to CWS and AiC. In this paper, we analyse the work of three groups of students (groups 1, 2, 3).

A POSTERIORI ANALYSIS

First step: Identifying dependencies through a paper model

Observation. This episode took place during the teaching session 1 after the students had experimented with a sheet of paper for some time. This is an excerpt from a discussion between two students of group 1 (S1, S2), two students of the adjacent group 2 (S3, S4) and the teacher (T).

76 S1: I think that the maximum water volume depends on the maximum volume of the metal sheet.

77 S3: [*Showing the cross-section of the gutter shown in the picture of the worksheet*] It depends on the area of this figure.

78 T: Why?

79 S3: Because the volume would be this [*showing the cross-section*] and the length of the whole thing [*gutter*]… (*does not matter*).

At this moment, S3 folds a sheet of paper with a very small vertical edge.

80 S3: What about if I fold the metal sheet like this? I want to have the maximum water volume.

81 T: The students here say that you need to maximise only the cross-sectional area.

82 S3: Actually, we need to concentrate on the changes. As the length grows the height diminishes and the area changes. So we have to find out a proportion for which the height and length will be exactly what we need.

83 S1: Basically, the maximum product.

CWS analysis. The episode reflects students’ work in the *discursive* dimension, when they try to make sense of the variations at a qualitative level. In addition to referring to concrete entities (water volume, “whole thing”), students introduce geometrical vocabulary as well as magnitudes (lengths, area, product...). The three vocabularies are “blended”: concrete objects, 2D and 3D geometrical elements, as well as lengths are not distinguished, and the magnitudes are not quantified. The students use words that make sense for them, but this use is not accurate enough to bring into play *semiotic* systems. Folding is used as an *instrument* to provide instances of the gutter adequate for the argumentation (gutter with a very small height), but insufficient to make precisely sense of the variations.

AiC analysis. The episode shows students *building-with* elements of the paper model, after *recognising* the opportunities offered by the sheet of paper working space for experimenting with different dimensions, and then *constructing* an understanding of the variations. When S3 folds with a very small vertical edge, she demonstrates that she *recognised* the opportunities offered by the sheet of paper to *build-with* the paper sides in order to (a) explain her idea of the relationship between the bottom and the vertical edge and their influence on the cross sectional area and then (b) *construct* her understanding of covariation. Later on, S1 abstracts the need for a ‘maximum product’ as a criterion for maximising the water flow (*constructing*, line 83).

Second step: Modelling in dynamic geometry

Observation. The episode is a dialogue between the teacher and students from different groups (teaching session 2). The following extract is taken from an interaction between the teacher and the group 2 students while working to construct the dynamic rectangle in the DG window (see Fig. 2 left).

67 T: You will need one point for the lower part of the gutter and one point which describes the maximum folding. Then we need another point between these two points to describe every time the different folding, but first of all we need to find the restriction of the construction.

68 S3: I propose to put point *D* in (0;0).

69 S4: We have to create a point *E* as (0;10) in case we fold the metal plate in the middle so that we get a segment for positioning the free point *C*.

After creating *C*, the students observed the folding in order to find an expression for the *x*-coordinate of *A*. Most of the groups attempted to find it through solving the equation x+x+y=20 for y. Students from different groups commented: “I tried to create *A*, as *(20-2*x ; 0)* but it did not work!”, “We created *A* as *(20-2*yC;0)* and it worked!”, “As for us, we created *A* as *(20-2*DC;0)* and it worked also!”.

CWS analysis. In order for students to take advantage of the DG possibilities they need to identify key elements of the model as geometrical objects using geometrical notations. The teacher’s intervention is crucial insisting on the choice of points defining a rectangle but the students also have their part: propositions for creating points *E* and *D* involving the constraints of the sheet, expression of the dependency of point *A* to point *C*. The work is in the *instrumental* dimension: adequate use of the instrument (DG) is at stake. The work also combines a *semiotic* dimension: students progressively integrate the notations of DG in order to express the *x*–coordinate of *A*. The proposition *20-2*x* by a student for this coordinate reflects an over interpretation of the sign “*x*” as denoting “any variable”.

AiC analysis. From an AiC point of view, we see here how the DG environment mediates students’ identification of the correspondence between geometrical objects and elements of the paper model. Building on their experience with folding the paper sheet, the students see the need to define a rectangle through four points and also to distinguish the point that ‘causes’ the dynamic change of the construction (*Recognising*). In order to organise the objects in the DG in a way consistent with the paper, the students make faulty and successful attempts to relate the coordinates of point *A* to measures dependant on point *C* (*building-with*). In the end, different symbolisations for the coordinates of point *A* are suggested by different groups (*constructing*) indicating students’ progressive coordination of their preceding sensual manipulation of the paper sheet with the available notation structures of DG.

Third step: Quantification of variations and distinction between dependent-independent variables

Observation. The selected episode refers to the experimentation of students in group 3 (S5, S6) with the geometric calculations tab of Casyopée (teaching session 3).

11 S5: Look at the area here [*pointing to the geometrical calculation DC x DA in the tab*]. We see that the maximum area is 50 and as we change this value... [*of DC*] … Okay. We cannot say that it is the maximum. If we change the point *C* in this straight line [*segment DE*] the area continuously decreases and maximises when it [*DC*] gets its maximum value.

12 S6: Look here [*in the geometric calculations tab*], it says 50 and we have the maximum value of segment *DC*. While we move down point *C*, we see that the area is decreasing too.

After that, the students tried the automatic modelling tab, choosing the geometrical calculation *DC x DA* as an independent variable and got an error message.

80 S6: It [*Casyopée*] cannot calculate a function.

81 T: Why? What do you think about it?

82 S5: It [*the geometrical calculation*] is dependent.

83 T: So, what?

84 S5: I will put here the area [*as dependent variable*] and here [*as independent one*] something that is independent of the rest of the others. That is, *DC*.

CWS analysis. In the *instrumental* dimension, the geometric calculation tab allows *continuous* observation of changes linking the movement of the free point to values of a quantity. There is a *semiotic* confusion when the students refer to the value of *DC* (DC = 5) for the area of 50 as the “maximum value of segment *DC*” since the maximum value of the length of this segment is actually 10. While students become aware of variations, they still miss the appropriate vocabulary to accurately describe them. The geometric calculation tab also provides feedback for distinguishing between independent and dependent variables. Making this distinction is not easy for students. They would prefer to consider the product *DC* x *DA* as the “main variable”. In the episode, the reason why Casyopée refuses this for the independent variable is not much discussed. The students switch to another choice that they justify allusively. It would be interesting to raise this point for discussion, making the *discursive* dimension more effective. Actually the product does not depend univocally of the position of C, and it would be useful that the students become aware that a given value of the product is obtained for two positions of *C* (e.g., the product is 32 for both lengths 2 and 8) as a difference with the length *DC*.

AiC analysis. By moving the point *C*, S5 *recognises* the potentialities offered by the tab to observe continuously changes in the numerical values of the relevant measures (line 11). Then, he is able to link specifically the two covarying magnitudes (line 11, *building-with*) to find a solution to the problem. Finally, S6 conceptualises the direction of the change of these magnitudes as an abstraction (line 12, *constructing*) stating that the area decreases as the length of *DC* decreases too. As regards the students’ work in the automatic modelling tab, initially they select the area of the rectangle *ABCD* (*DC* x *DA*) as independent variable. The provided feedback (*recognising*, line 80) leads them focus on the features of the selected variable and to select correctly the pair independent-dependent variables (*building–with*, line 82). Finally, they conceptualise *DC* as a coherent element ‘generating’ the change of the area of the rectangle *ABCD* in a rather primitive sense (“independent of the rest of the others”) (*constructing*, line 84).

Fourth step: Expressing covariation through the use of variables

In teaching session 4, students of group 1 observe the changes in each column of the table of values of the function they obtained by “automatic modelling” in the symbolic window.

41 S1: From the table we see the maximum value at 5…

42 S2: It shows the area for each value that *x* takes with the restrictions we set.

43 S1: If we change the step it shows us the area in relation to the side *DC* that changes by 0.5. We see that 5 remains the value of the side *DC* so as to have the maximum area. We notice that for the different values of *x* the area changes and reaches its maximum in *DC* [*equal to 5*]

44 S3: Wait. For the various values of x, the area changes and finds a maximum for *x* = 5 with the area equal to f(5) = 50.

CWS analysis. In the *instrumental* dimension, the students coordinate observations on the table (algebraic window) and values of quantities in the geometrical calculations tab. They experiment with a smaller step (0.5) and observe a behaviour of the function consistent with their previous observation. In the *semiotic dimension*, quantities (length of sides) co-exist with algebraic symbols (x, f(…)). We observe a connection between quantities and algebra potentially productive for making sense of symbolism. While in the preceding episode students referred to the value of *DC* for the area of 50 as the “maximum value of segment *DC*”, here they correctly refer to it as “the value so as to have the maximum area”. It seems that working on the dependency as a mathematical function facilitates students’ conception of variables.

AiC analysis. Here the students move in the direction of working with mathematical variables representing quantities that they encountered in the preceding model of the situation. They observe the variation of *DC* and its value that maximises the area *ABCD*. By linking *DC* with column *x* of the table values (line 42), S2 helps S1 to conceptualise the variation of *DC* as the variation of the independent variable *x* (line 43, *recognising*). Then, S1 experiments with different values in the step of the table so as to determine the maximum area (line 43, *building-with*). Finally, S3 conceptualises function as covariation by relating the changes in the two columns of the table in terms of dependent and independent variables (line 44, *constructing*). Function appears as a model for a covarying pair of variables and students work with it through a standard tabular register.

Connections

The above analysis reveals the progressive character of the connections students develop through building-with and constructing actions (AiC approach) in the different working spaces (CWS approach). At the second step, the students engage in the work of folding and recognise key functionalities of the DG system in order to model a variable rectangle conforming to constraints of the paper model (*“we fold in the middle…”)* and to express these constraints with the adequate notation. At the third step, the students associate a moving point in the DG figure with variations of measures (*“If we change the point C…”*), before focussing on the covariation of these measures independently of the figure. At the fourth step, the students make connections in the instrumental and semiotic dimensions between the measure space and the mathematical function space. This helps students recognise mathematical variables as representing quantities and make sense of the variations of the function as representing covariation of measures.

This is evidence that at each step, students first connect a new working space to the former (*recognsing*) and then develop conceptualisations (*building-with* and *constructing*) inside the new working space. Beyond this step-by-step abstraction, the students in our study were able to make wider connections: for instance, a group of students commented on their solution by considering a table of the function (mathematical function working space) together with notions that exist in other working spaces (area, side…): “*We see in the table that the area is maximised when the coordinate of the free point C [i.e. yC] is 5. That is, we have the maximum area when one side [of the gutter] is half of the other.”*

Summarising, in order to address students’ conceptualisation of function as covariation in different settings, we considered a situation based on an authentic modelling task that utilises the potential of different models, corresponding working spaces and their connections for creating meaningful abstraction processes. The transcribed episodes were subjected to a dual-lens a posteriori analyses that complement each other:

- CWS allows (a) specifying the working spaces involved in approaching functions and modelling and their connections and (b) analysing and coordinating a triplet of dimensions of students' mathematical work in these spaces (semiotic, instrumental, discursive).
- AiC offers a fine-grained analysis of the students' constructing processes within and between these spaces.

DISCUSSION AND CONCLUSION

By networking CWS and AiC, we introduced a framework to make sense of the implemention of an authentic modelling learning situation and of students' conceptualisation of function as covariation. Below, we summarise how the two frameworks combine first in the a priori analyses and then in each one of the four steps.

In the a priori analysis, CWS provides an analytic description of the means offered to students for working on a model at each step of the situation and assumptions about connections that students can make between the models. AiC builds on this analytic description in order to identify precisely the elements of knowledge at stake and makes hypotheses about how students will learn through the connections, that is, which constructs might be observed during the process of knowledge construction. Then the combination is productive since CWS alone allows merely minimal assumptions regarding learning, but it provides a reliable structure for more precise hypotheses by AiC. In the first step, CWS insists on the instrumental dimension of the work on the paper sheet and AiC shows how this work can be described in terms of recognising, building-with and preparing to construct a first understanding of co-variation. In the second step, CWS and AiC both use their own constructs, but are consistent in noting how the students use their previous experience of the paper model to build a DG model. AiC goes deeper into knowledge construction, while CWS insists on the coordination of the instrumental and semiotic dimensions in students' work. In the third step, CWS points out difficulties for working on measures in both the semiotic and in the discursive dimension, while AiC shows how the students progressed in their understanding of covariation in spite of these difficulties. In the fourth step, CWS notes that students connect quantities and algebraic entities in the instrumental and semiotic dimensions while AiC shows how this connection allows them to understand function as covariation with specific techniques and notations. As a whole, at each step, we observe processes of abstraction that at first connect a new working space to the former and then develop conceptualisations inside the new working space.

With regard to the research questions, the combination allows appreciating the potential of this implementation of an authentic modelling problem in which students engage thoroughly and consider successive models from a concrete mock-up to a mathematical function. Implementation of similar tasks based on experimental situations can be informed by the insights provided by the present work in many ways. For instance, the a priori analyses may support teachers to orient students' activity during the task enactment in the classroom by identifying different working spaces. Also, the dual-lens analysis may further sensitise task designers, teachers and researchers to (a) the diversity of models and working spaces, (b) the intended constructs in these spaces, and (c) the complexity of students' constructing processes in developing the targeted conceptualisations.

As regards the contribution of each theory, the CWS allows distinguishing the different spaces involved in the complex path from authentic context to algebra and their connections. It also allows focusing on three dimensions (instrumental, semiotic and discursive) and their coordination in students' work. The RBC model of AiC is powerful here to make sense of students' progress, but could not be put into operation without the structure provided by CWS. To sum up, CWS is useful in providing ("horizontally") a plurality of settings (physical context, geometry, measures, algebra) in terms of signs, instruments and modes of reasoning for students, whilst AiC offers concepts and expected strategies and an account of knowledge construction ("vertically") within and between these settings. This takes the analysis further and allows a deeper look at students' cognitive evolution during knowledge construction. CWS builds on the MWS theory. As we pointed out, it emphasises the interest of offering students a variety of working spaces and the opportunity of making connections. It combines well with AiC, which helps to figure out how connections *between* working spaces contribute to conceptualisation. In the MWS theory, conceptualisation is considered through specific "geneses" *inside* a MWS linking an epistemological and a cognitive level. In this paper, thanks to AiC, we also observed conceptualisation *inside* each of the working spaces. Thus, a further step of networking could study what aspects of conceptualisation are better addressed by AiC and by geneses and possible mutual benefits.

REFERENCES

Artigue, M., Bartolini-Bussi, M., Dreyfus, T., Gray, E., & Prediger, S. (2006). Different theoretical perspectives and approaches in research in mathematics education. In M. Bosch (Ed.), *Proceedings CERME 4* (pp. 1239-1244). Sant Feliu de Guíxols, Spain: FUNDEMI IQS – Universitat Ramon Llull and ERME.

Dreyfus, T., Hershkowitz, R., & Schwarz, B.B. (2015). The nested epistemic actions model for Abstraction in Context: Theory as methodological tool and methodological tool as theory. In A. Bikner-Ahsbahs, C. Knipping & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 185–217). Dordrecht, The Netherlands: Springer.

Douady, R. (1986). Jeux de cadres et dialectique outil–objet. *Recherches en didactique des Mathématiques,* 7.2, 5–31.

Duval, R. (2006). Quelle sémiotique pour l'analyse de l'activité et des productions mathématiques ? *Relime, Numero Especial,* 45-81.

Kaiser, G., & Schwarz, B. (2010). Authentic modelling problems in mathematics education. Examples and experiences. *Journal for Didactics of Mathematics, 31*(1), 51-76.

Kuzniak, A., & Richard, P. R. (2014). Spaces for mathematical work: Viewpoints and perspectives. *Relime*, *17*(4.1), 17-27.

Lagrange, J.-B. (1999). Complex calculators in the classroom: theoretical and practical reflections on teaching pre-calculus. *International Journal of Computers for Mathematical Learning*, *4*(1), 51–81.

Lagrange, J.-B. (2018). Connected working spaces: designing and evaluating modelling based teaching situations. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of PME 42* (Vol. 3, pp. 291-298). Umeå, PME.

Lagrange, J.-B. & Psycharis, G. (2014). Investigating the potential of computer environments for the teaching and learning of functions: A double analysis from two research traditions. *Technology, Knowledge and Learning*, *19*(3), 255-286.

Minh, T. K. & Lagrange, J. B. (2016). Connected functional working spaces: a framework for the teaching and learning of functions at upper secondary level. *ZDM—The International Journal on Mathematics Education*, *48*(6), 793-807.

Robert, A., & Vandebrouck, F. (2014). Proximités-en-acte mises en jeu en classe par les enseignants du secondaire et ZPD des élèves: analyses de séances sur des tâches complexes. *Recherches en didactique des mathématiques*, *43*(2–3), 239–285.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.