FUNCTIONS IN TECHNOLOGICAL ENVIRONMENTS: FROM MULTI-REPRESENTATIONS TO CONNECTED FUNCTIONAL WORKSPACES

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This paper questions the idea of function on the basis of classroom research at upper secondary level in the Casyopée project (Lagrange, 2010). In this project, students are introduced to functions by working on the same functional dependency in four settings. The underlying theoretical framework is a multi-representation view of functions. In cross-experiments and cross-studies this approach has been confronted with other approaches. This implied to reconsider the idea of function and to examine the potential of the idea of connected functional workspaces for designing and analysing classroom situations around functions especially in technological environments.

Keywords: Casyopée, Modelling Cycle, Functions, Functional Workspaces, Semiotic systems.

This paper examines the potential offered by technological environments and associated signs systems in the mathematical work of the student around the notion of function. It does this on the basis of classroom research at upper secondary level, supported by the design and experimentation of the Casyopée software (Lagrange 2010).

Activities on functions mobilize many material or mental artefacts associated with varied semiotic systems. We consider these artefacts as representations, that is to say, entities with which the mathematician or apprentice mathematician interacts and to which he or she assigns a meaning. Semiotic systems are special entities systematizing practices in a given culture.

The work around Casyopée fits into the functional approach to algebra as encouraged by many curricula, while relying on didactical frameworks, especially around the semiotic systems (Duval 1999, Radford 1999), and around functions viewed as co-variations (Thompson 1994). The activities offered to students are considered through a "modelling cycle" where the same functional dependency is studied in four settings: (1) a physical system (2) a dynamic geometry construction modelling the system (3) co- variation between quantities involved in the construction (4) a function defined by a symbolic expression (Minh 2011).

In cross-experiments and cross-studies (Lagrange & Psycharis 2013; Lagrange & Kynigos 2014) this approach has been confronted with other approaches showing that the modelling cycle is adapted to the analysis of situations beyond the work around Casyopée, but also the limits of considering this work in a multi-representation approach. This opens questions discussed in the paper:

- Why and how stay away from a Platonic view of the notion of function?

- What are the benefits of seeing the four settings of the modelling cycle as connected functional workspaces?

In the conclusion, I discuss further work especially on cognition inside and between workspaces.

HELPING STUDENTS TO MAKE SENSE OF MATHEMATICAL SYMBOLISM WITH THE SUPPORT OF THE COMPUTER

Comparing two research projects aiming at using technology to offer students "more learnable" mathematics, Hoyles, Lagrange and Noss (2006) stressed how, in the paper/pencil context, symbolism is a major obstacle for most students.

> *Paper/pencil algebraic infrastructures make it necessary for individuals to pay considerable attention to manipulation, and key mathematical topics are only amenable to those who had already been inducted into fluent algebraic representations and calculations. This means that many never engage with the mathematical topic at hand* (p. 269).

My main concern at this time was whether, knowing these difficulties and the potentialities of dynamic technologies like dynamic geometry, spreadsheet, etc. for exploration of mathematical notions and problems, mathematical symbolism would survive outside pure professional mathematics. I noted that

> *Computer symbolic systems (CAS), have been presented as a means to overcome students' difficulties in paper/pencil manipulations (…) Although promising, this approach has not been, in our view, sufficiently discussed from an epistemological point of view, and its 'viability', that is to say the conditions in which it could be effective in actual classrooms, remains problematic* (p. 270).

I explained that the main aim of the Casyopée project I was starting with colleagues and teachers, was to address the challenge of developing a CAS tool that could be effective in secondary classrooms. Relatively to representation, this development was mainly inspired by Duval (1999)'s framework presented in the next section and exploited for discussing a task for students motivated by this approach.

A « REGISTER » APPROACH OF MULTI-REPRESENTATION

For Duval (1999, p.4), "there is no other ways of gaining access to the **mathematical objects** but to produce **some semiotic representations** (…) On the other hand, the understanding of mathematics requires not confusing the mathematical objects with the used representation*s*". According to Duval, rather than opposing material and mental representations, it is necessary to distinguish two types of implementation of representations by the subject, the automatic activation of an artifact (material or mental) that he names organic system, and the intentional exploitation of a semiotic system (Figure 1).

Implicitly, Duval presupposes the existence of mathematical objects. Then in automatic activation, this object itself produces the representation, for instance "proportionality" produces points in a line on a concrete graph, and in intentional use, a semiotic system, that is to say a set of symbols and rules for using them, represents the object (that is to say is used in place of the object). Thus, representing "something" can be thought of in two ways: the activation of a mental or physical device making this "something" work or the conscious use of a semiotic system in place of the "something".

Automatic	**Intentional**
Activating an organic system or a physical device	Bringing into play a semiotic system
The representation is the outcome of a physical action of the	The representation denotes the represented object

Figure 1. representations and their implementations (from Duval 1999)

For Duval, each semiotic system or "register of representation" has its own specific means of representations and processing. Duval stresses the need for a specific focus of teaching learning on the processes of work inside the registers (treatments) *and* between the registers (conversion). Relatively to functions at upper secondary level, two registers are generally involved, the register of symbolic expressions and the graphic register of curves of functions. Many research studies show that in classroom practices the register of symbolic expressions dominates, and in this register rote treatments dominate over interpretation of expressions and understanding of the rules and purpose of transformations[1]. Conversions are always from mathematical expression to graphs, occurring in standardized tasks like finding the true graph of a function from its symbolic expression.

In this section, I look at the conditions in which technology can help a more versatile use of each of these registers as well as a balance and coordination between these registers, drawing from a classroom situation using Casyopée, studied by Minh (2011)It deals with pre-calculus students and the following task.

Consider the family of functions f_k *over the set of real numbers* $\mathrm{f}_k(x) = x \cdot e^{-x} + k \cdot x$ *,* ***k*** *being a real parameter. Study the variations of* f_k *depending on* ***k****.*

[1] For a recent example, showing how even when using a software like GeoGebra for introducing a function, a teacher does not break with this dominant conception of functions, see Robert & Vandebrouck (2014, p. 264).

This task has been chosen because a pure symbolic solution is not directly accessible through standard techniques, since the root of the derivative cannot be calculated symbolically. An effective standard technique is to study the variations of the derivative, after calculating the second derivative, but this technique is not known by students. Another feature is that the function has three different behaviours depending on *k* (see Figure 2) one on the left for *k* greater than a certain value (e^{-2}), the second between this value and zero, and the third for k negative.

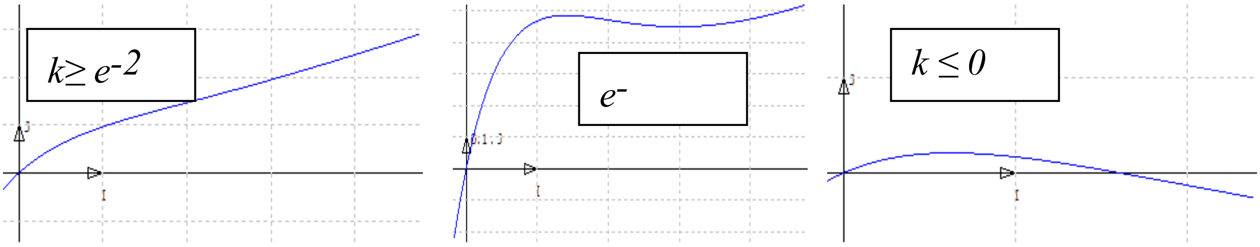

Figure 1. Three behaviours of the function

The important thing is that this value is close to zero, and thus a pure graphical exploration might overlook the intermediate behaviour. It can be expected that most students will first animate the parameter for integer values and then skip this intermediate behaviour. However a close inspection of the graph of the derivative for *k=0* and *k=1* (Figure 3) will give a clue that 'something happens' between these values: passing from 0 to 1, the curve is translated vertically, and then an intermediate value would bring the curve to intersect the x-axis in two points. This idea of a translation is confirmed by the symbolic expression $f'(x)=(-x)\cdot e^{-x}+e^{-x}+k$ where *k* is an additive constant. Then, in order to carry out the task, the students have to perform non-routine processes inside the symbolic and the graphic registers, *and* to coordinate explorations and observations in the two registers.

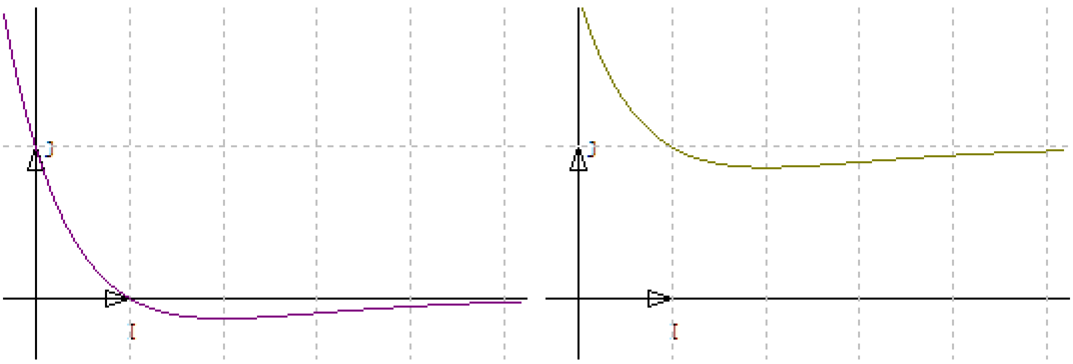

Figure 2. The derivative for k=0 and k=1

Designed as a CAS system especially devoted to functions at this level, Casyopée allows students to perform automatically standard calculation like derivatives and solutions of equations, as well as to graph functions. Piloting the values of *k* by way of a slider helps to animate the graph of the function, and also to instantiate this parameter inside expressions of the functions. In short Casyopéee is expected to give security for calculation and to favour joint interpretation of graphs and symbolic expressions.

Investigating students' solutions at various stages, we observed three types of answers.

In a first type 1 (Figure 3), the intermediate behaviour is not detected. The student keeps the default value 1 for the step in the animation of the parameter and does not closely look at the derivative. This type of answers is frequent by beginners not yet acquainted with the management of the parameters in Casyopée and who do have not a flexible use of the derivative.

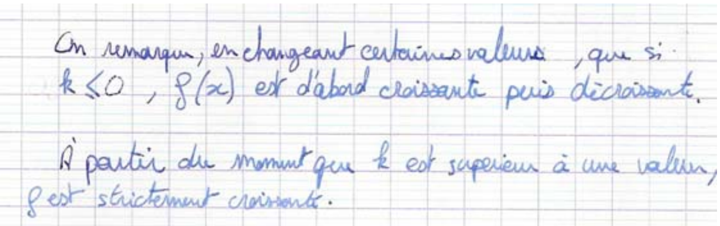	By changing some values, I observe that if $k \leq 0$, f(x) is first increasing then decreasing. After *k* is higher than a certain value, *f* is strictly increasing.

Figure 3. A type 1 answer

In a second type of answer (Figure 5) the student realizes that it is easier to observe the graph of the derivative as compared to the graph of the function. Animating the graph with a smaller step of the parameter, she detects the intermediate behaviour, but she dwells in the graphic register, and then she cannot find the critical value (Figure 5).

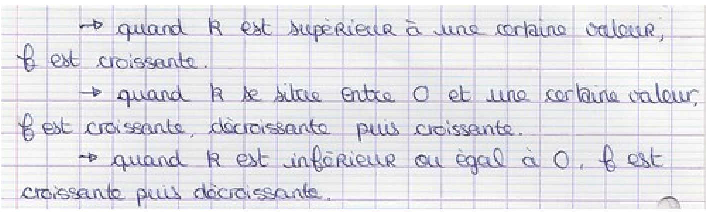	• When *k* is higher than a certain value, *f* is increasing. • When *k* is between 0 and a certain value, *f* is, decreasing, then increasing. • When *k* is lower or equal to 0, *f* is increasing then decreasing.

Figure 4. A type 2 answer

In a third type of answer (Figure 6), the student finds the critical value e^{-2} by asking Casyopée for the solution of $f'(2)=0$ (Figure 5). Analysing her approach, we identify four steps:

- she observes on the graph that the derivative is always maximum for x=2,

- coordinating the symbolic form of the derivative and its graphic representation, she notes that *k* is an additive constant and then the animation translates the graph vertically,
- she figures out that to find the critical value, she has to “bring the graph” onto the x-axis, which is not possible by animating for decimal values of the parameter,
- she solves symbolically by way of the equation in *k*, f’(2)=0.

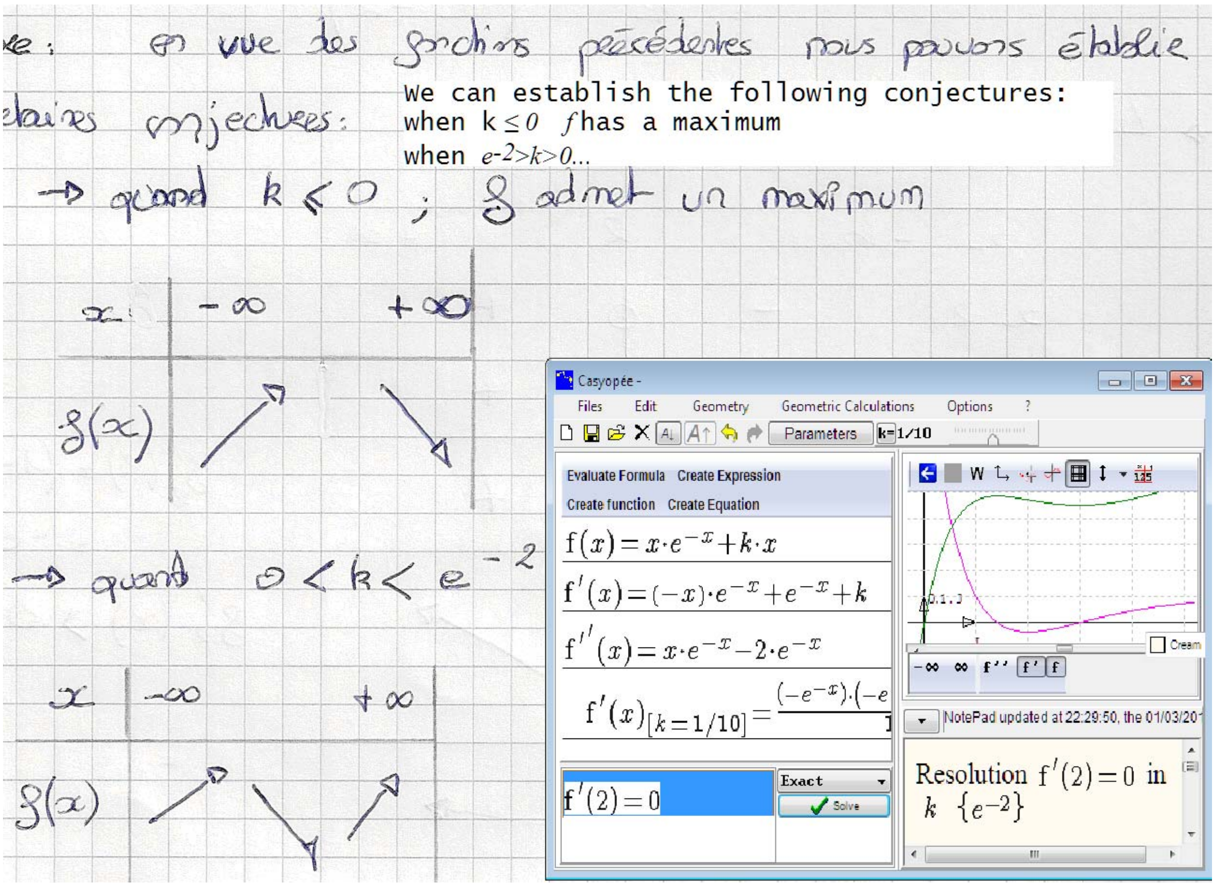

Figure 5. A type 3 answer

This third type of answer is observed by students mature with the use of Casyopée (Minh, ibid). It is evidence of a very effective coordination between the symbolic and the graphic register. Casyopée provides means that make this coordination effective. It does not in itself give the students this capacity for coordinating registers. This capacity is the result of a process of students’ growing awareness of Casyopée’s functionalities *and* of the idea of functions as expressed in several registers. This is a process of *instrumental genesis* (Lagrange 1999) that Minh (2011) observed by students all along two years of studying pre-calculus.

ALTERNATIVE REPRESENTATIONS OF FUNCTIONS

Duval’s framework is very rich and many researchers draw on, especially when dealing with digital representations. As said before, it presupposes that “ideal” mathematical objects (notions, ideas, concepts…) exist. As shown above, Duval's registers of representation were productive in the development of a tool for helping students deal with tasks about functions. However, it was only a first stage and I regretted in Hoyles, Lagrange and Noss (2006) that

> *Casyopée works on functions given by symbolic representations and till now provides no enactive representation for non-symbolic functional exploration* and *limitations in the students' experimental activity, especially when modelling phenomena.* (p. 282)

I also pointed out the need to develop features allowing students to work with enactive non- symbolic representations. Short after writing this, I had the opportunity to work with colleagues on an extension of Casyopée in a project whose name was "Representing Mathematics with Digital Medias" (Lagrange & Kynigos 2014). With regard to Casyopée, one goal was to explore new, non standard representations of functions. We had also to carry on "cross studies" of digital tools developed inside the project. In this section I report first on new ways of representing functions introduced in this extension of Casyopée, and then on other views coming from the "cross studies".

Modelling dependencies in a physical system

One outcome of the work on Casyopée in ReMath and after was a broadened view on functions in the students' mathematical activity. The idea of modelling dependencies in the physical world was thought of as a way to allow students to work with enactive non-algebraic representations of phenomena and to pass fluidly between algebraic and non-algebraic representations. More precisely, the Casyopée team imagined a "modelling cycle" organising models of a dependency in specific settings. Figure 7 illustrates this cycle with the example of the "amusement park ride" that I will use in this sub-section.

Basically, dependency is a human experience involving items in a *physical device*. In this setting, one item moves and another follows. I consider *dynamic geometry* as a second settings and the product of an evolution: human beings used drawings since the prehistoric times to "represent" features of the physical world and they made them evolve to allow accurate work related to physical devices; recently *dynamic geometry* added the particular capacity of "representing" dependencies: when the user drags an object on the screen, another object follows on the screen. Passing from a physical device to a dynamic geometry figure is in no way trivial, since it implies the identification of relevant features of the device and of their geometrical dependencies.

Quantities constitute a third setting for working about dependencies. According to Thompson (2011, p.37), a *quantity is* the outcome of a process of "conceptualizing an object and an attribute of it so that the attribute has a unit of measure". Thompson (ibid.) stresses the role played by quantitative reasoning in students' operations of generalizing, and of the role that generalizing plays in students' development of algebraic reasoning. He offers a number of '*dispositions'* that can aid students' construction of algebraic reasoning from quantitative reasoning. Among these, consistent with the school level the Casyopée project is intended for, I retain reasoning with *magnitudes*, that is to say attributes of objects considered independently of the unit in which they are measured. As Thompson (1994) says "one way to think of the evolution of today's many ways to think of functions is as the current state of a long battle to conceptualize our world quantitatively and then this world of quantities or magnitudes, often overlooked in secondary curricula, is a key representational setting for students' access to functions. Especially, in this setting, distinguishing functional dependency from co-variation, and identifying an independent variable are crucial challenges. As a fourth setting*, functions defined by a symbolic expression*, like those considered in the previous section, is the outcome of a historical process of emergence of functions as an individualized mathematical entity by mathematicians like Descartes, Newton and Leibniz as a tool for building infinitesimal calculus.

The extension of Casyopée

The extension of Casyopée carried out in the ReMath was intended to provide an environment supporting these various representations and their connections. It consisted in adding first a dynamic geometry window and second, representations of measures and of their covariation in a "geometric calculation tab". The existing part of Casyopée remained in an enhanced form, under the denomination of "symbolic window". The "geometric calculation tab" was designed as a unique feature, allowing to explore covariations between couples of magnitudes, to export couples that are in functional dependency into the symbolic window and then to define a function modelling the dependency, likely to be treated with all the available tools. In order to help students in modelling dependencies, this can be done automatically (Lagrange 2010). We will refer to this functionality as "automatic modelling" below[2].

For the Casyopée team, the goal was that students meet functions from meaningful questions and under connected representations in these four settings. An example is the "amusement park ride" situation.

A wheel rotates with uniform motion around its horizontal axis. A rope is attached at a point on the circumference and passes through a fixed guide. A car is hanging at the other end.

The motion is chosen in order that a person placed in the car feel a smooth transition at high point and abrupt at low point and then, in the classroom, the focus is on the identification of these different transitions on a physical model *and* on a mathematical

[2] For a detailed presentation of this functionality and Casyopée's other capabilities, please see the documentation downloadable from the web site. http://www.casyopee.eu/

function modeling the movement of the car. It is expected that students associate the transition at the highest position of the car with a turning point of the function and the transition at the lowest position with the non differentiability of the function at the corresponding point. The dynamic figure and the expression of the movement as a dependency between magnitudes are essential parts of the modelling activity, helping to make sense of the relationship between the physical model and the mathematical function.

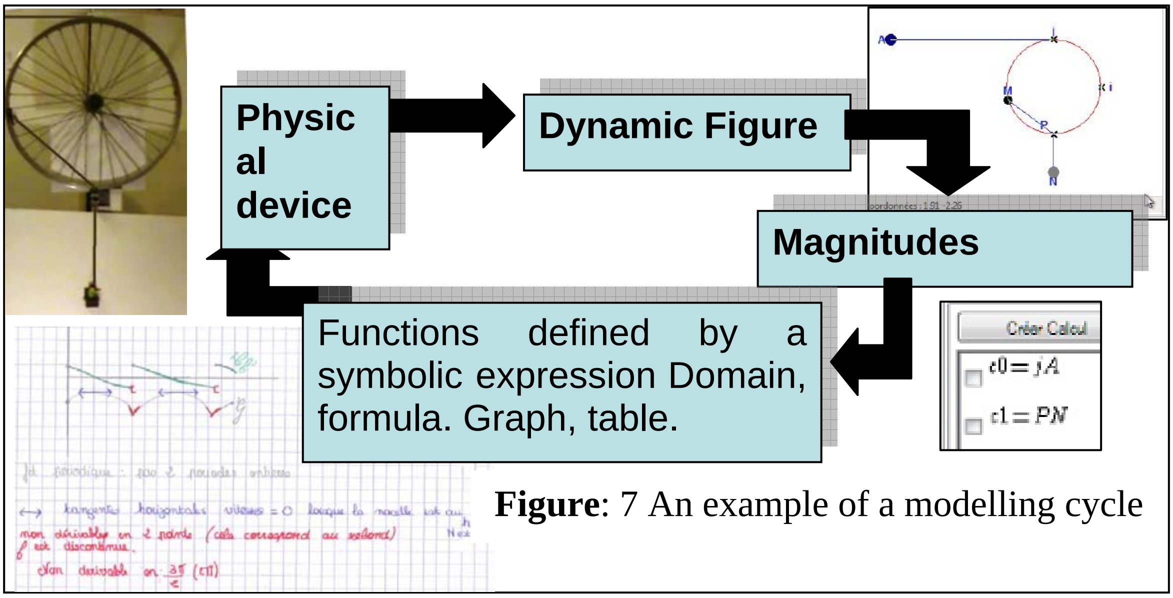

Figure: 7 An example of a modelling cycle

A classroom observation

The situation was carried out with a 12th grade class (17-18 years old students) in a 90 mn session. The problem was given in “real life” settings, the students being able to manipulate a scaled device. The students were first asked to describe to describe what is happening at the lower point and whether it is different as compared to the high point. They were encouraged to refer at how a person in the car would ‘feel’ the difference and to draw a graph to explain their answer. Then they were told that, during the session, they have to build a mathematical model in order to better investigate the phenomenon.

The first step of modelling consisted in building a dynamic geometry figure in Casyopée, replicating the device. The following indications were given to the students: the rope is attached to the wheel in a mobile point M and the guide is on the fixed point P. The car is in N (figure 7). The wheel is supposed to be put into rotation by pulling on a horizontal rope jA. This implies not trivial constructions for the point M in order that the circular distance IM is equal to the linear distance Aj, and for the point N in order that MP+PN is constant. The students had to use Casyopée first to implement a dependency between magnitudes modelling the dependency in the figure, then to obtain a function defined by a symbolic expression thanks to “automatic modelling”, to get the derivative and should notice and finally to identify precisely the points of non differentiability.

I report on this situation in five steps: (1) the students' spontaneous model of the physical situation (2) how they built a dynamic geometry model (3) how they choose the dependant and independent variables and how they interpreted this choice (4) how they worked on the function obtained via Casyopee's automatic modelling (5) how their understanding of the physical situation progressed after working on the symbolic expression of the function.

Students' spontaneous model

At the beginning of the session, after presenting the situation and demonstrating by animating the scaled model, the teacher asked the students to describe what is happening at the lower point and whether it is different as compared to the high point. Figure 8 illustrates a typical answer. Students said that at the high point, the car stops and they had some difficulties to explain what is happening at the lower point. The more common expression, drop shot, is not accurate because it means that the car is arriving at a certain speed, stops and starts up again at a lower speed. Students illustrated by a graph of a piecewise linear function. Actually they thought that because the wheel rotates uniformly, the car's movement should be piecewise uniform.

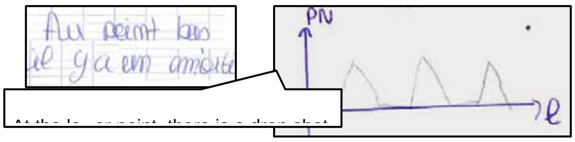

Figure 8: Students' spontaneous model

Building a dynamic geometry model

This was a difficult part. Students' poor practical knowledge in trigonometry explains why they needed help to define M in order that the circular distance IM equalled the linear distance Aj. It seems more surprising that they found difficult to define N in order to make MP+PN=2 (the length of the rope). After the teacher indicated that PN is known when MP is known, some students used a circle centred in P with a radius of 2-MP and defined N as an intersection point with the y-axis, and others directly defined N with the coordinates
$(0; y_P-(2-MP))$.

Choosing the variables

Generally the students had no difficulties to operate the choice with the software. However, their expression was sometimes confused when explaining the choice. For instance a pair of students wrote in the report: "We choose distance Aj as the (independent) variable" and added "Aj is a function of the coordinates of N".

Working on the function obtained via Casyopee's automatic modelling

The students obtained the derivative by using Casyopée's automatic modelling under the form $x \rightarrow \frac{-\cos x}{\sqrt{2\cdot\sin x+2}}$. Casyopée issued warnings because this function is not defined everywhere. Students ignored the warnings and obtained a graph with wrong vertical segments (pink in Figure 9).

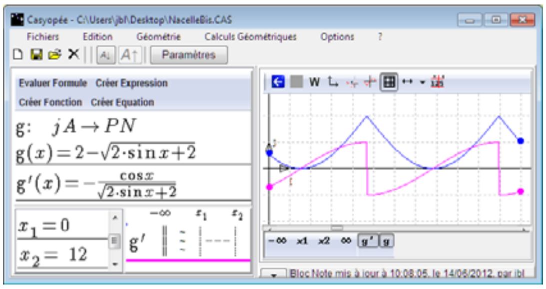

Figure 9: A graph of the derivative with wrong vertical segments

The teacher drew students' attention on these segments and they recognised that there should be discontinuities of the derivative corresponding to the low points. The teacher asked them to compute the position of these discontinuities. No students did this from the formal definition of the derivative. They rather came back to the physical device or to the dynamic geometry figure, looking for the value of jA corresponding to the lower point of the car. After they found these values and excluded them from the definition of the derivative, they got a correct graph (Figure 10).

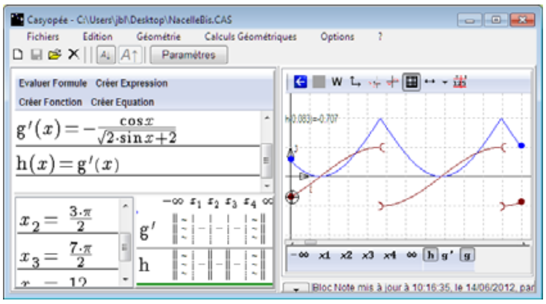

Figure 10: A correct graph of the derivative

Students' understanding of the physical situation after working on the function obtained via automatic modelling

Students' understanding was much better in the last phase. They identified the derivative and the car's speed, saying that the speed is null at the high point corresponding to a horizontal tangent on the graph of the movement. Implicitly, they recognised that at the lower point the car starts up again briskly at the same speed, speaking of "rebound" corresponding to non differentiability points, rather than of "drop shot" implying softer stop and restart. This is evidence that the students connected features of the mathematical function with a phenomenon experienced in the physical word. Clearly students got a broad view on this function, coordinating many aspects or representations, without dominance of one representation. The symbolic formula was an aspect between others, the pole of the derivative giving a specific insight, complementary to the discontinuity on the graph, and the feeling of "rebound" at the lower position of the car.

Other insight coming from a Remath "cross study"

The experimentation of several situations based on the above 'modelling cycle' comforted the team in the productivity of the enlarged view on functions on which Casyopée's extension is based. A "cross study" carried out in ReMath provided additional unexpected insight. ReMath aimed at investigating contextual elements influencing research about math education and technology, and to progress towards theoretical integration. A method in Remath was "cross studies": each piece of software developed in the project was "cross-experimented" by two teams and this cross experimentation was followed by a "cross-case analysis" (Lagrange & Kynigos 2014), one team being in charge of the design and development of the piece of software. Here I consider the Cruislet cross study that associated a Greek team and the team developing Casyopée.

The Greek team was influenced by a constructionist framework. They designed Cruislet as a navigation microworld in which users direct planes across the Greek geography by issuing navigation instructions in either graphical/Cartesian or spherical/polar coordinate systems, in direct stepwise mode or by way of Logo programming. The modalities of employing exposed by the Cruislet team were consistent with their theoretical perspective. For instance, the team implemented a "guess my flight" situation based upon the use by students of a procedure that made one plane perform a flight to an arbitrary position while another reached a dependent position, each of its coordinates being a linear function of the same coordinate of the first one. Using this procedure first as a black box and then decoding the procedure in order to propose similar challenges, students could make sense of the situation by investigating the co-variation of the planes and conceiving the first plane's position as an independent variable and the second plane's position as a dependent variable.

The Casyopée team encountered difficulties to make sense of Cruislet's features for educational task, and especially did not recognize the potential of a situation like "guess my flight" to allow students to approach function, a central domain of interest for the Casyopée team. Again the Casyopée team's view of function was not broad enough: the domain of co-variation at stake in Casyopée is 2D geometry and the functions are polynomial, rational or transcendental one variable real functions rather than multilinear functions. It helped to see that it would be pointless and counter-productive to consider *a* notion of function transcending all possible representations.

Functions: a broad plurality of representations rather than a single concept

Duval's idea of considering several register of representations, stressing the necessity of educating students both in the treatments inside registers and in the conversion between registers was productive in the first phase of development of Casyopée, since it helped to build a tool and situations of use breaking with current narrow tasks and dominant conceptions of functions.

However, given the big variety of representations that we had to deal with in the extension of Casyopée and in the cross-studies, I came to consider unproductive the Platonic view of function as a single mathematical object, acting in different settings and denoted by all these dissimilar representations. Actually, the idea of analytic function, as invented by Descartes, Newton and Leibnitz dominates at secondary level, and teachers' and students' conceptions of functions are strictly dependent on an symbolic view of functions. It means that even when they consider situations where geometrical or physical dependencies are involved, these dependencies are used as a "faire-valoir" (complementary or contrasting character in a play) rather than actual functions. There is then a danger that in the implementation of situations like the "amusement park ride" above, too much emphasize is put on symbolic facts to the detriment of the other interpretations[3].

It seems then more productive to think of functions as a constellation or a network of many representations, each rooted in a particular practice, no one dominating and with more or less strong connections between them rather than connections to a single object. This seems consistent with Radford's (1999) view that a « representation does not appear as a pointer to an abstract idea but as the contextual instantiation of social modes of knowing as expressed and contained in signs". Especially considering the case of functions, Radford critics a narrow view of functions and stresses that linking representations cannot be made without a common context.

> *The learning of functions is seen as the capability of a student to move from one representation to another (tables, graphics, algebraic formulas, etc.). These "external" representations are seen as the expression of the same concept —that of function. In my view, each semiotic system (tables, graphics, etc.) leads to a particular concept (...) What this means is that a contextualization among representations will be needed to link them, and this requires a different pedagogical action. (p. 149)*

Mathematical and functional workspaces

Above, I described how it is possible to think of functions as a plurality of representations, each related to a specific setting. It is important to reflect on these settings, since a representation does not exist alone and as Radford (1999) says, being acquainted with representations and making links between these requires a contextualisation. For me, a representation can be seen as a tool used with other artefacts to work in a specific domain or workspace. Mathematical workspaces (Kuzniak & Richard 2014) generalize geometrical workspaces. A workspace is what

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[3] See footnote 1 for an example.

allows individuals to work on a mathematical task. In a workspace signs or language have their own existence as specific artefacts, rather than as "representations" of ideal objects. For students, the work aims both at knowing better the objects (what the problem is about), at mastering the artefacts (material and symbolic) and at using wisely a relevant framework of reference (i.e. a set of properties allowing explanation, deduction or inferences).

It is important for mathematics educators to organize mathematical workspaces offered to their students, paying attention to (1) the internal relationships between the three poles, objects, artefacts and framework of reference inside a workspace and (2) the means they provide to students for connecting these workspaces. Retrospectively the work carried out about functions in the Casyopée project and illustrated by the above example can be thought of as an instance of this organization, including the design of specific (technological) artefacts insuring internal consistency and easing the connection between workspaces. Table 1 recapitulates the different poles for the workspaces involved. Since the physical device cannot be qualified of "mathematical", the noun functional workspace will be used for all settings.

Poles\workspaces	Physical device	Dynamic figure	Magnitudes	Algebra and calculus
Object	Mechanical dependency	Co-variation of geometrical objects	Co-variation of measure, variables	Mathematical Functions
Artefacts	Device, language	Primitives of construction, dynamicity, geometrical language	Language, specific symbolic expressions	Mathematical symbolism and language
Framework of reference	Mechanical constraints	Geometrical properties	Quantification	Algebra and calculus theorems

Table 1: Four workspaces

FUNCTIONAL WORKSPACES

After introducing workspaces as a way to make sense of representations in the Casyopée project, and as a means to break with a Platonist view of functions, the goal of this section is to extend the reflection to another similar research study, in order to test the relevance of this introduction on a wider basis.

The research question in the "enlarging-shrinking alphabet" study (Lagrange, Psycharis 2013) was how a computer environment could help students to understand dependencies and express them in formal notation. It is then consistent with Casyopée's aim. However it has been conducted at a different school level (6th grade), with a different software environment, and functions considered here are linear rather than polynomial or transcendental. Turtleworlds, the computer environment, was chosen in order to prompt students to construct relationships and figures according to the rules of proportionality *and* to allow formal expression of these relationships. It consists of three components: Canvas, Logo and Variation Tool. The elements of a geometrical construction in the canvas are expressed by way of a Logo procedure. After a procedure depending on variables is defined and executed with a specific value, the Variation Tool provides a slider for each variable. For instance, a correct procedure to draw a 'letterN' could be:

to letterN :r fd :r rt 135 fd :r*1.41 lt 135 fd :r end[4]

Since, in the above example, the procedure properly expresses the proportional relationship between the sides of the letter, dragging the :t slider enlarges or shrinks the letter while conserving the shape.

The task was to construct enlarging-shrinking models of capital letters with one variable corresponding to the height of the respective letter. In a first phase, the students were given a procedure with two variables

to letterN :r :t fd :r rt 135 fd :t lt 135 fd :r end

The students were encouraged to explore with various values of :r and :t using both sliders. In a second phase they were invited to change into a one variable procedure, drawing a correct letter for all values of :r. In a third phase they had to look for a general 'method' to prevent distortions applicable to other angles.

Lagrange & Psycharis (2013) report on an observation of two students (A. and C.) in the second phase. They considered the relation of the two values as "200 plus forward 40" which they used in the construction of the respective original pattern of N. Thus, they substituted the variable :t with the functional expression (:r+40). When A. moved the slider: r to the value 220, the figure was distorted and C. conjectured directly for the changes in the functional expression ("we need to add 50"). This way students continued to work with additive relations (e.g., :r+50, :r+45) but moved from constant differences to adjusted differences according to the different values of the independent variable. However, successive dragging on the variation tool confirmed that the use of an additive algebraic expression constituted an erroneous strategy for constructing an enlarging-shrinking model of N holding for "all values of :r". This is how students moved to a multiplicative strategy under the supervison of a researcher (R.)

[4] 1.41 is approximatively |1/cos(135°)|, fd : forward, lt : left turn, rt rigth turn.

R. : Since the one [i.e. the vertical segment] is 100 and the other one [i.e. the slanted segment] is 145. What is the relation between 145 and 100?

C.: It [i.e. the slanted length] is neither half. It is half … let's say plus 45.

Researcher: Half?

C.: Not exactly. [After a while] One and 45 which we have already put here [i.e. the expression :r+45].

R.: If this is one time and half bigger than the tilted one. How much is it?

A.: :r plus one and a half.

C.: [to A.] No, it would be two times and a half then. It's :r plus half. It is one time and half bigger.

A.: :r plus half.

C.: [Thinks for a while] :r times one and a half. Lets' try it [C. then goes straight to the Logo editor and types the linear relation as 1.5*:r for the slanted length.]

After experimentation with changing the values of the functional operator they accepted as value 1.42.

The observation shows that the functional relationship expressing proportionality is really at stake here, the students struggling to give up with additive relationship, at first with ambiguous utterances, then correcting in a fractional expression (one and a half), translating into a multiplicative relationship (1.5*:r) and finally adjusting thanks to the feedback of the figure. The specific challenge is to formulate a dependency between the length of the vertical segments and the length of the slanted segment in a formal form allowing its expression in a Logo procedure.

Students work in four functional workspaces similar to those identified in the Casyopee experiment above. The physical system of reference is an (evocated) letter conserving its proportions at different sizes. The geometric figure is a path of the turtle in three segments with a given angle between them. In phase 1, the path depends on two variables and in the next phases, the challenge is to program the path in order that it depends on one variable while conforming to the goal that it represents the letter N. Magnitudes like angles and lengths are involved. An algebraic expression occurs to express the dependency inside the LOGO procedure.

While activating the sliders and working on the Logo procedure, the students consider together the physical system, the geometrical figure, the dependency between magnitudes and an algebraic expression of this dependency. Their task is actually to understand the constraints of the physical system as a dependency linking two magnitudes and to find an expression for this dependency in order to write a procedure using a single variable.

Here also, physical situation, dynamic figure, magnitudes and formalism offer four different workspaces. As a difference with the Casyopée experiment above, the progression of functional meanings is not seen as a progression through the settings: there is no cycle. Students have constantly to jump for one workspace to the other. Similarly, there is no one to one correspondence between the settings and the three components of the computer environment. As a difference with Dynamic Geometry, the figure can be manipulated only by way of sliders (of the variation tool) that act on measures. The Logo procedures can be viewed as functions associating the figure to one or more variables, kinetically controlled via the sliders. However, the procedures do not express directly the functional relationship between lengths which is at stake. The actual expression of this relationship is within the procedure, embedded in commands for the turtle.

Distinguishing four similar workspaces in the two experiments helps to evaluate choices made in the design of both experiments and software. In Casyopee, starting from a purely algebraic environment, a dynamic geometry window has been appended, and linking algebra and geometry implied conceiving a space to work on magnitudes. Physical systems were considered in order to take advantage of functional dependencies experimented in the physical world and « embodied » in human cognition. This is a rather epistemological view of the workspaces' organisation. The Turtleworlds experiment is based on the constructionist aim to help students access « meaningful » algebraic forms. This is rather a cognitive view that does not clearly distinguish the spaces in which students have to work. Comparing the two experiments, one can say that in Casyopée, on the one hand the workspaces are well identified, but on the other hand the activity is somewhat rigidly organised as a (circular) path along the workspaces. In the Turtleworlds experiment, the activity is less constrained and more motivated, but there is a chance that students never clearly identify the objects, artefacts and frameworks of reference in which they work: for instance, here it is not clear what the letter :r represents, a magnitude or an algebraic variable.

CONCLUSION

This paper offered connected workspaces as an alternative to multi-representation for the domain of functions. This view departs from a Platonist conception of functions as a single ideal object. It helps also to recognize representations not as entities existing for themselves, but rather as artefacts co-existing with others artefacts helping to work on specific objects and controlled by specific frameworks of reference. This is productive to appreciate and contrast experiments involving computer environments. In this paper, reflecting on functional workspaces was done retrospectively, in order to make sense of experiments and software designed with another agenda in mind. I expect that the idea of connected functional workspaces will in the future, also help task and software design, as well as a priori analysis.

Cognition is another topic for further reflection. How do students working in functional workspaces access to meaningful views about functions? Kuzniak &

Richard (2014) propose to add a cognitive layer, connected to a workspace by specific geneses, one being the genesis of mathematical instruments introduced by researchers like Lagrange (1999) under the inspiration of cognitive ergonomics. For me, the question at stake is how students working on specific objects with specific artefacts develop an understanding of a specific framework of reference. I would then consider that in each workspace, an instrumental genesis is at stake, constructing "instruments" from material artefacts (technological or not), as well as from other semiotic artefacts (discursive or figurative). Especially in the domain of functions, linking the instruments built in geneses of a plurality of workspaces is important. Presmeg (2006) offered to think of semiotic chains as a way to bridge a gap between activities of everyday practices and mathematical activities and to organize a process in which "goals, discourse patterns, and use of terms and symbols, all move towards that of classroom mathematical practices in a way that has the potential to preserve essential structure and some of the meanings of the original activity." Clearly, this kind of process is at work in both "amusement park ride" and "enlarging-shrinking alphabet" situations above. Characterising these processes as a way to connect workspaces is a promising direction.

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