CONNECTED WORKING SPACES FOR SECONDARY STUDENTS' UNDERSTANDING OF CALCULUS: MODELLING A SUSPENSION BRIDGE THROUGH "JIGSAW" GROUP WORK

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This paper is about the design of classroom situations of modelling real world situations at upper secondary level to confront student to questions and ideas in calculus. At this level, curricula avoid formal approaches of calculus notions, but in current strategies, modelling is used as a motivation for algebraic methods, rather than as a basis for scientific work. The paper proposes the framework of connected working spaces for designing and implementing sustainable situations of modelling helping a wider approach of the curriculum in these classes, and giving sense to the concepts taught by making students understand interactions between these concepts and the interactions between these concepts and other sciences or real world situations. The example of a situation of modelling a suspension bridge is presented on the basis of four working spaces (statics, geometrical, algorithmic, and mathematical functions). The implementation is carried out through "jigsaw group work" that helps students work in these spaces and make connections between them.

Keywords: modelling, suspension bridge connected working spaces, calculus, Casyopée, jigsaw group work.

As a researcher, I am working with teachers in the Casyopée group in the context of the French national curriculum at upper secondary level. Like in other regions of the world, a large part of this curriculum deals with functions. Formal aspects, like the structure of the number line or a definition of limits are not considered and then tasks for students are supposed to favor intuitive approaches of notions. However, while emphasizing problem solving in various domains, the curriculum actually privileges application of classical “algebraic” methods. Real world situations are considered in a narrow approach of modeling: “translating into the mathematical language”. Classroom practices reflect this narrow approach: the teacher invites students to express “informal conjectures”, and then, abruptly requests them to express and prove these conjectures by pure mathematical means (Minh and Lagrange, 2016). The consequence is an over emphasis on algebraic forms and manipulation, not helping students to make sense of notions in calculus.

In reaction to this narrow treatment of functions and calculus, our[1] objective is to promote secondary calculus as an effective tool to understand the world, rather than a meaningless manipulation of symbols. There is a big emphasis on real world situations and on modelling in mathematics education. Conspicuous examples are the 14th ICMI Study, Applications and modelling in mathematics education (Blum, Galbraith, Henn & Niss, 2002 and 2007) and the two 2006 special issues of ZDM - The International Journal on Mathematics Education (volumes 38(2) and 38(3)). Modelling is however a very open domain, with a lot of different options. One option is to consider students' modelling activity as a way to develop modelling competencies independently of the learning of specific mathematical knowledge (Mass, 2006). Another option is to create modelling situations focused on a given knowledge or concept, through "guided reinvention" (Gravemeijer & Doorman, 1999). In addition, besides all the progress made in mathematics education research, authors like Burkhardt (2008) pointed out the difficult existence of mathematical modelling practices in ordinary classrooms.

Our approach is specific in the sense that we are looking for situations (1) that are neither purely oriented towards modelling competencies, nor towards the "reinvention" of an isolated mathematical concept, (2) that are sustainable in ordinary upper secondary classes and (3) that help a wider approach of the curriculum in these classes, giving sense to the concepts taught by making students understand interactions between these concepts and the interactions between these concepts and other sciences or real world situations. More precisely, the goal of the Casyopée group is to create and experiment secondary classroom situations where students understand functions and calculus, through convergent approaches of a question about a phenomenon. In parallel, the group develops a geometric and algebraic software environment (Casyopée) as a tool to study these phenomena. Examples of phenomena considered as a basis for classroom situations are:

- Games of “chase and prey” (will Coyote hit BipBip?); it is based on the common experience that mobiles moving on convergent trajectories may or may not collide, depending on their respective speeds. In a situation inspired by Cazes and Vandebrouck (2014) Coyote and BipBip have given speeds, and students have to adjust the angle of Coyote's trajectory in order that collision occurs. It brings to the forth a fundamental not obvious idea: in any model, both movements have to be parametrized by the same quantity, the time.
- Closeness on a curve (what is the position on a curve closest to a given point outside this curve?). The question is how to express closeness and how to parametrize the movement of the mobile in order to get an accurate model.

[1] "We" and "our" refer to the Casyopée group. I am grateful to R Halbert, C. Le Bihan, B. Le Feuvre and M. C. Manens who introduced me to "jigsaw group work", and carried out the design and implementation of classroom situations after my proposition about modelling bridges.

- Vertical movement in an amusement park ride (how to give account of different behaviors at low and high points). In a situation, a person is in a car subject to a periodic vertical movement. She experiences a "smooth" transition at high points and an "abrupt" transition at low points. This can be related to students' bodily experience, and explained by way of models of the mechanical device animating the car, including a mathematical function whose derivative is discontinuous for values corresponding to low points.
- Tension in the cable of a suspension bridge (how does the tension evolve along the bridge, and how does this explain the shape of the cable?). Students study a suspension bridge, and have to explain why the main cables are not horizontal strait lines. By concrete experiments, they have to realize how a weight suspended on the cable influences its shape, and understand the idea of tension as a vector quantity evolving along the cable, because of the weight of the deck considered as a collection of infinitesimal units.

Thus the question for the Casyopée group is how to implement classroom situations taking advantage of real world, while confronting students to questions and ideas in calculus. In the next section, this question will be refined by way of a theoretical framework, then we will develop the last example.

THEORETICAL FRAMEWORK

I am currently building a framework inspired by Kuzniak (2013) and Kuzniak & Richard (2013). Lagrange (2015) introduced the idea of connected functional working spaces as an alternative to multi-representation for the domain of functions. This helps to depart from a Platonist conception of functions as a single ideal object. It helps also to recognize representations not as entities existing for themselves, but rather as artefacts co-existing with others artefacts helping to work on specific objects and controlled by specific frameworks of reference. The example of the amusement park ride helped to present four different working spaces offering specific means for modelling a dependency between measures. One is related to the physical device, the second to a dynamic figure modeling this device, the third is about magnitudes in dependency and the fourth is characterized by the classical mathematical symbolism and notions in calculus. The situation was based on the "embodied cognition" hypothesis and on the use of Casyopée's functionalities. The "embodied cognition" hypothesis is that students' reference to bodily activity in physical settings and to emotions experienced in this activity, can be a basis for deeper understanding of notions in calculus, as compared to a pure formal approach of these notions (Rasmussen et al, 2004). Casyopée especially helped to build a link between the geometrical dependency and a function modelling this dependency.

The idea of connected functional working spaces was further developed by Minh & Lagrange (2016). We were influenced by Kuzniak (2013) that presents geometrical working spaces as a way to avoid misunderstandings in geometrical education, for instance with regard to how students should reason, developing their spatial intuition and ability with instruments, or rejecting these in favor of formal deduction. Like geometrical working spaces, considering functional working spaces allows teachers and students to work on functions in various spaces, including spaces where functions are experienced without algebraic formalization, avoiding the predominance of a working space restricted to algebraic representations and manipulations. Like with geometrical working spaces, working in a specific functional space should allow working on functions with specific instruments and under control of specific rules. We specified a dynamic geometry space, a measure space, and an algebra space, and indicated specific functionalities of Casyopée bringing artefacts in each space and means for connecting the spaces. We examined then the functionality of this framework for implementing and analyzing classroom situations and for analyzing students' and teachers' evolution relatively to functions, in terms of geneses related to each space. We took the above example of "closeness on a curve" to consider students' activity in each space and connections between spaces, in order to develop principles of design for a classroom situation, to assess their efficiency and to highlight differences with current classroom situations based on geometrical optimization where the teacher sees exploration as a "motivation" and not a specific work bringing specific conceptualizations.

This paper develops further this idea of connected working spaces as a framework to design and evaluate situations of modelling, allowing students to understand functions and calculus, through convergent approaches of a question about a phenomenon. The idea is that the study of a question involves several working spaces around objects, each object in a space being a model[2] of the corresponding object in another. Each space is characterized by the artifacts that allow work on the question and a reference framework or theoretical framework. This implies that the question makes sense to work in different spaces and gives meaning to objects, without favoring a space.

QUESTION AND METHOD

Kuzniak & Richard (2013) stress that working spaces are not given, but are constructed in the teaching learning process. Thus, working spaces of reference may exist around socially accepted standard for formulating questions and answers organized by favoring certain artifacts and modes of thought. But they have to be converted and organized to become "suitable" working spaces in a given educational institution with a defined function. As Kuzniak (2011), said "les experts concepteurs de la réorganisation didactique des diverses composantes de l'espace de travail (…) aménagent un ETM qui peut être idoine parce qu'il respecte les intentions et le cahier des charges de l'institution demandeuse ". We wish to play, even modestly, the role of experts as understood by Kuzniak (2011), and therefore our current questioning is:

[2] In contrast to the narrow view of a model as "a real world situation translated into the mathematical language", we think of models as more or less tangible systems in relation of similarity or representativeness, and helping to understand a complex phenomenon, without hierarchical organization.

> How to organize suitable working spaces that make students understand interactions between concepts taught at upper secondary level, and the interactions between these concepts and other sciences or real world situations? What conceptualizations can be expected in these working spaces, and how these conceptualizations interact with more standard mathematics?

In addition, we wish to try the above mentioned “embodied cognition” hypothesis mentioned above and explained by Núñez, Edwards and Matos (1999) by stressing that learning and cognition cannot be fully understood without considering the shared biology and fundamental bodily experiences of human beings and concluding that mathematics education should provide a learning environment “in which mathematical ideas are taught and discussed with all their human embodied and social features".

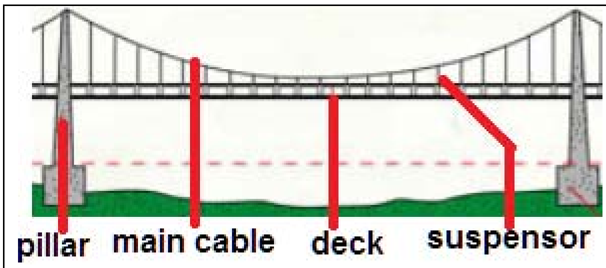
Looking for a situation to implement the above idea of interaction between concepts in calculus and involving real world and other science, we thought that studying a suspension bridge would be a suitable basis, for the reasons explained below.

Suspension bridges

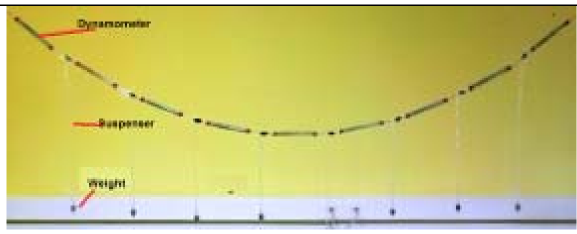
A suspension bridge is a type of bridge in which the deck (generally a roadway) is hung below suspension (or main) cables by vertical suspensors equally spaced. There is no compression in the deck and this allows a light construction and a long span (figure 1a). The weight of the deck applied via the suspensors results in a tension in the main cables. The main cables are anchored on top of pillars and the pillars support the compression resulting of the tension. As stated above, the central question is to find models of a main cable, allowing to solve technical questions like the value of the tension and therefore of the compression in the pillars for given data characterizing the bridge.

A discrete model derives from the finite number of suspensors: a main cable is represented as a collection of segments, beginning and ending at the anchoring points, and separated by the suspension points linking the main cable and the suspensors. Modelling the tension in one of the main cable can be made by considering the sequence of tensions $\vec{T_i}$ along every segment, and for each, the values of the horizontal component and of the vertical component $(\mathrm{H_i};\mathrm{V_i})$. The static equilibrium law, applied at every suspension point, implies that the horizontal component is the same in all segments. It also implies that the sequence of values of the vertical component is in arithmetic progression, With a choice of the direction of the tension, the vertical component is negative at one anchoring point, with an absolute value equal to the half of the weight of the deck supported by the cable (a quarter of the whole weight in case of two cables) and positive at the other one with the same absolute value, and the common difference is the value ΔP of the weight of a portion of the deck supported by a suspensor (figure 1c).

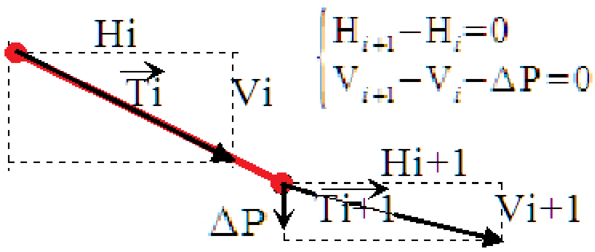
In the same discrete model, the slope of a segment is the ratio of the vertical by the horizontal components of the tension in this segment, and therefore is also in arithmetical progression. Knowing the position of one anchoring point (top of a pillar) it is possible to compute the sequence of the coordinates of the points of the collection of segments modelling the main cable, and to adjust the value of the constant horizontal component in order to get the desired elevation of the center of the cable above the deck (figure 1e).



a) A suspension bridge



b) A physical model of a suspension bridge



c) Tensions on a segment (red) of the discrete model.

```
//coordinates of the anchoring point
x ← −640
y ← 163
//Tension on the anchoring point
V ← −5
Pour i allant de 1 à n
    x ← x + 1280/n
    y ← y + 1280/n · (V/H)
    V ← V + 10/(n−1)
Fin Pour
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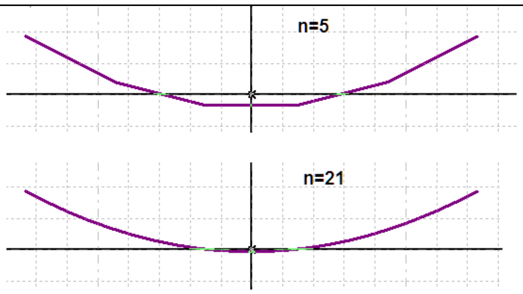
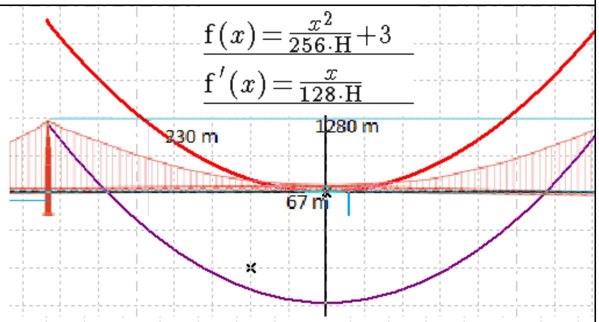
d) An algorithmic discrete model of the suspension points; n is the number of segments, and H (resp. V) the horizontal (resp. vertical) component of the tension.



e) The discrete model for two values of the number of segments



f) Equation and curve (red) of the continuous model, compared to a picture of the bridge and to the discrete model (magenta). The horizontal component H has to be adjusted in order that the three models fit.

Figure 1: models of a suspension bridge

Given the big number of suspensors, one can look for a curve, limit of the collection of segments modeling the cable when this number tends to infinity, and the distance between suspensors tends to zero. Since the limit of the slope of a segment whose extremities tends to a point, is the gradient of the curve at this point, the arithmetical nature of the sequence of the slopes of the segments implies that the gradient depends linearly on the position on the curve, and, by integration, the curve is of quadratic nature (i.e. an arc of a parabola). This is the continuous model (figure 1f).

Affordances, constraints and general design

The brief presentation above shows that studying a suspension bridge implies considering data in the real world as well as a number of interrelated concepts in physics and calculus: tension, static equilibrium of forces, projection of vectors, slope of segments and gradient of curves, arithmetic progression and linear function, integration, discrete and continuous models, limits and integration… All these contents are taught in secondary curricula, thus the goal for students is not to "reinvent" each of them in isolation, but rather to recognize how a question in a real world situation involves understanding these concepts operationally and in interaction. In the French curriculum, the study of a suspension bridge can be carried out in the last year of the secondary scientific stream (12^{th} grade, Terminale). In the previous year, students studied arithmetic progressions and derivative of functions (in connection with the slope of tangent lines), and learnt to program the values of sequences, as well as to program approximate curves of functions whose derivative is known (the Euler method). In this last year, the students learnt about tensions in physics and about integration in mathematics.

A constraint results of a character of these classes: students pass an important exam at the end of the year, evaluating standard proficiencies rather that deep understanding. Thus there is limited time for situations going beyond isolated proficiencies in typical tasks, and a teacher has to highlight the contribution of less standard tasks in these situations.

The Casyopée group used the framework of connected working spaces to design a classroom situation exploiting the potential of the study of a suspension bridge. We consider four working spaces. In the first one, the object at stake is the sequence of tensions at the connection points of the suspensors, and the rules are the static equilibrium law and the properties of arithmetic progressions. Artefacts are concrete measurement devices used in physics and mathematics, dynamometers, angle protractor, and also more "abstract" tools like the decomposition of tensions in vertical and horizontal components. We name this working space, the static systems working space, or shortly, the statics working space. The second working space deals with the discrete model of a main cable and then with geometrical objects. The main rule is the analytical definition of a segment: students have to compute the coordinates of the end point, knowing the coordinates of the other point, the slope and the difference between abscissas. We name this working space, the geometrical space. In the third working space, an important artefact is Casyopée's programming environment, that allows computing a series of points by way of a simple iterative treatment (figure 1d); the points allows defining a continuous piecewise functions, whose graph, in the case of the bridge, represents the discrete model of the main cable (figure 1e). We name this working space, the algorithmic space[3]. Finally, the objects in the fourth space are functions governed by classical rules in calculus. This is the mathematical functions working space. Integrating a linear function should not be difficult for students. However, the formula of the linear function involves a parameter (the horizontal component H of the tension), and then students might be uncomfortable and benefit of Casyopée's symbolic capabilities. They can also use Casyopée to get a curve of this continuous model and compare to a picture of the bridge and to the discrete model, and adjust the horizontal component H in order that the three models fit. Thus Casyopée brings artefacts, useful in the geometrical algorithmic space and in the mathematical functions space, and also helping to connect these.

IMPLEMENTATION

As said before, there are constraints at this pre-exam level and the implementation is limited to three and a half hours and organized in four phases. The first phase is one hour long and has been prepared with the physics teacher. It aims first to introduce students to questions related to bridges, particularly suspension bridges. They are invited to consult a dedicated website (http://structurae.info/ouvrages/ponts-et-viaducs), to select and sketch four bridges of different types, to look at a video illustrating the idea of tension along a horizontal rope and the fact that, whatever the tension, the rope is no more a straight line, as soon as force is applied vertically on a point. They have to answer three questions: (1) why in a suspension bridge the main cables are not horizontal? (2) what type of functions do you propose to model the main cable? (3) is the shape of a main cable determined by the length of the suspensors ? Also in this first phase, the students have to build two apparatuses like in figure 2, read the tensions in the dynamometers and the angles, compute the horizontal and the vertical components of the tensions and verify the static equilibrium of forces.

Figure 2: apparatuses in the first phase

The second phase is 50 mn long. At the beginning, the data related to the Golden Gate Bridge is presented to the whole class. Students also look at a physical model (figure 1b), and one student reads the tensions in the dynamometers to the whole class. Then students are split into groups of four. Each group has one task, A or B, C or D.

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[3] About algorithmic working spaces, see Laval (2015).

Task A is related to the statics working space: inspired by the work in the first phase, students have to consider the sequence of horizontal and the vertical components of tensions at the connection points, recognize that the horizontal component is constant and compute a formula for the series of vertical components.

Task B is related to the geometrical algorithmic working space. A formula for the value of the slope of each segment in a discrete model of the main cable is given to the students, depending on a parameter H, and on the number n of segments. Students have to compute formulas for generating the series of *x* and *y*-coordinates of the suspension points.

Task C is related to the algorithmic working space. An algorithm like in figure 1d is given to them; they have to enter and execute the algorithm in Casyopée, interpret the parameter n, and adjust the parameter H in order that the model given by the algorithm conforms to the shape of the cable (figure 1e).

Task D is related to the mathematical functions working space. Students have to search for a function *f* whose curve models a main cable (continuous model). They are informed that the horizontal component of the tension in the cable is a constant H and that, in chosen axis, the formula for the vertical component of the tension at a point of x-coordinate *x* is given by the formula $V(x)=\frac{P\times x}{2L}$, where P is the weight of the deck and L its length. They have to find a formula for the derivative of *f*, taking into account that the tension is in the direction of the tangent to the curve. Then, using Casyopée, they have to find a formula for *f* and adjust the parameter H in order that the curve of the function *f* conforms to the shape of the cable (figure 1f).

The third phase is also 50mn long. The students form new groups, also of four. Each of these new groups is made in order to bring together one or two students of each of the previous groups respectively doing task A, B and C. The groups are invited to share their findings and to write a report emphasizing the important points of the study. In this work a student coming for instance from a group that did task A is an expert in the statics working space, informing the students that did other tasks, of the methods and results in this working space. This organization in two series of groups is inspired by the « Jigsaw Classroom », which is designed as

> a cooperative learning technique that reduces racial conflict among school children, promotes better learning, improves student motivation, and increases enjoyment of the learning experience… Just as in a jigsaw puzzle, each piece — each student's part — is essential for the completion and full understanding of the final product[4].

The advantage of this organization is that students, in a period of time compatible with the constraints at this level, get a global understanding of the study of a problem, performing by themselves some of the key tasks related to this problem, even when

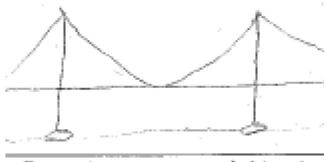
[4] https://www.jigsaw.org/

they do not 'solve' the problem in all aspects. As illustrated above, the idea of several working spaces for the study of a problem is a guide for organizing the tasks.

The fourth phase (30 mn long) is a collective synthesis led by Professor.

OBSERVATION AND EVALUATION

This implementation was observed in a class of 35 students by the end of March. This class was familiar with the "jigsaw classroom" organization that had already been put into operation for collaborative work on a lesson. The students were mostly average achievers. The contents at stake in physics and mathematics had been taught to students in previous lessons. The phases have been video recorded, and interviews were conducted with 3 students after phase 3. Due to the limited size of this contribution we restrict our analysis to some hints about how students behaved and progressively understood the structure of the bridge.

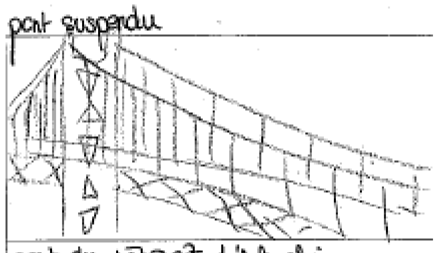


a) Pont du détroi d'Akashi

a) Pourquoi dans un pont suspendu les câbles porteurs ne sont -ils pas horizontaux ?

Le tablier appuit, et pousse le câble vers le bas (en tension)

c) Pour un pont suspendu, est-ce que la forme des câbles porteurs est due à la taille des suspentes ?



b) pont du détroit d'Akashi.

a) Pourquoi dans un pont suspendu les câbles porteurs ne sont -ils pas horizontaux ?

Le tablier va pousser le cable vers le bas.

c) Pour un pont suspendu, est-ce que la forme des câbles porteurs est due à la taille des suspentes ?

Non, la forme est due aux forces exercées sur le cable.

Figure 3: answers in the first phase

In the first phase, most students sketched a suspension bridge without suspensors (figure 3a). They generally explained the non-horizontality of the main cable by the weight of the deck, but agreed with the false explanation of the shape by the length of the suspensors. They understood from the video that the weight of the deck "bends" the cable, but they did not link the shape of the cable with the uniform repartition of the weight, thanks to the suspensors. Isolated students showed a better understanding, writing that the shape is the consequence of the forces (or tensions) on the cable (figure 3b). In the rest of the first phase, after overcoming instrumental difficulties with the apparatuses (figure 2), the students correctly recorded the angles and the intensity of forces. They recognized the static equilibrium law, but they generally did not compute the components. Especially the fact that the horizontal component is the same on all the three dynamometers in the apparatus with two weights (figure 2) did not appear. In the classroom discussion, before dividing the class in groups for phase 2, the teacher insisted on the decomposition of a vector in components and on the variations of the components of the tension in the cable at stake in the next phases. He also repeated that the goal is to study mathematically the shape of the cable.

For the group work in phases 2 and 3, I report only on a series of four groups observed doing each task in phase 2, and on one group in phase 3 bringing together students observed in phase 2, leaving for further work a comparison to other students in the whole class. In phase 2, students observed doing task A mainly succeeded, while difficulties were observed for students doing other tasks. Students doing task B started by sketching a bridge with a lot of suspensors, not allowing to consider segments. They were prompted by the observer to limit to 4 suspensors. They took time to find the coordinates of the anchoring point, and had difficulties to use the formula given for the slope of the segments and the distance between suspensors in order to calculate the coordinates of the next point. Actually, this calculation involves several parameters related to data of the bridge, a common situation in physical sciences but not in mathematics. Students doing task C took time to enter the algorithm in Casyopée. Nothing or a wrong display appeared on the screen, because of small mistakes. They could correct only when the observer helped them to analyze the algorithm. They identified the parameter n as related to the number of suspensors and proposed the value 83 (the number of suspensors in the Golden Gate Bridge). They considered that this value is "close to infinity" and that is why the curve did not appear as a collection of segments, in contrast to small values of n. When the observer explained that H is a tension, they get aware that increasing the value of this parameter "straightens" the cable, and found a suitable value. Students doing task D found a formula for the vertical tension, but had difficulty to interpret the fact that the tension is in the direction of the tangent to the curve.

In the group of phase 3, each student explained her task and her work in the preceding phase. The video recording shows that other students listened attentively and asked for further explanation. The parameter H was identified by students as playing a role in each task; for instance when a student who did task C did not remember the effect of increasing H, confusing with the "height of the cable", the student who did task A corrected him, saying that it is a tension and then increasing should "straighten" rather than "slacken" the cable. The same student helped to overcome the difficulty met by the student who did task D to find the direction of the tangent to the curve and then the derivative of the function, saying "you just integrate the quotient of V and H". Going further in task D, the students were confused by the parameter H in the denominator, some proposing a Ln(H) in the antiderivative. They could achieve task D, using Casyopée. In contrast, the unfinished task B, and the connection with the algorithm in task C were not discussed.

Three students were interviewed after phase 3, as a method to further evaluate what connections students made between working spaces during the group work. They stressed that the situation was more complex than usually ("we had to connect a lot of different things") and that they were "not used to mix physic and mathematics". Commenting the first phase, they showed how their awareness of the structure of a bridge progressed: they mentioned the role of the suspensors and made the link between a suspension bridge and an arched bridge with regard on how the deck is supported. They made also the link between the apparatus with two weights and a suspension bridge "with two suspensors". They still had difficulties in considering the slopes of the segments in task B in order to find the coordinates of the suspension points. However, they correctly interpreted the algorithm of task C, and were able to connect the evolution of H, and *x* and *y* respectively to task A and B. They did not show clear awareness that the function of task D was a limit of the continuous piecewise function of Task C. From graphical evidence they thought that it was more or less the same function for big values of n. Visualization is then the way students connected the algorithmic and the mathematical function working spaces. The observer asked to explain why the gradient in a point of the curve is the quotient of V and H. The expected answer was that the tension has the direction of the tangent, but the students simply wrote $f\,'(x) = \Delta y/\Delta x = V(x)/\,H$ without more explanation. It seems that the first equality is common in the physics course, and that the second derives from the definition of the components in task A. Thus students made a connection between the statics space and the mathematical function space without explicit consideration of a limit.

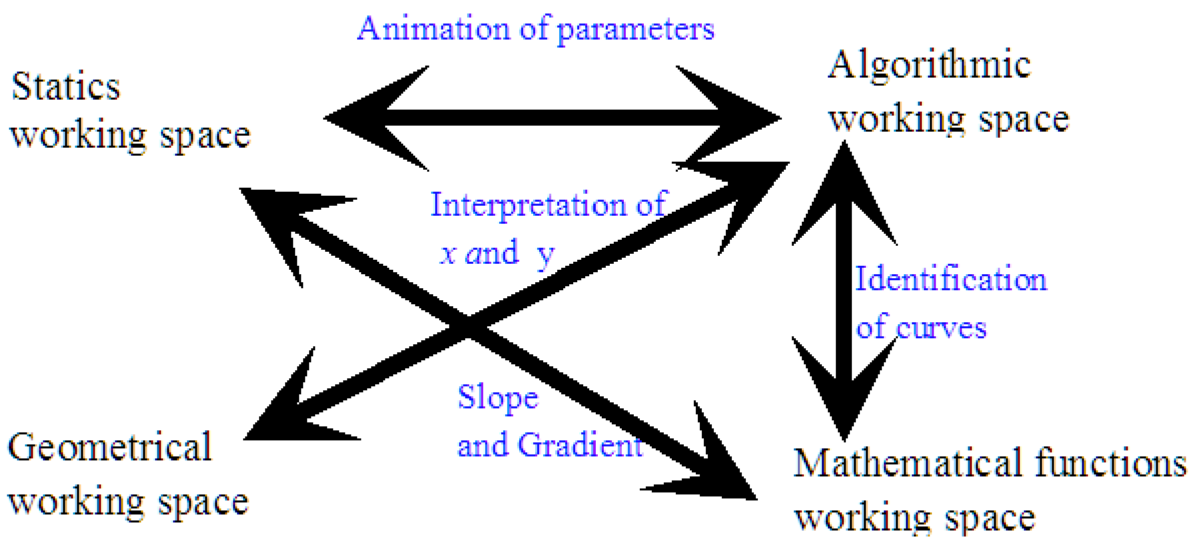


Figure 4: the connections made by students between working spaces

CONCLUSION

In Minh and Lagrange (2016), we demonstrated the need for considering three connected working spaces when implementing a suitable situation about geometrical optimization, taking the example of closeness on a curve, as evocated in the introduction. The knowledge at stake was the awareness of functions as models of dependencies, and the necessity of quantifying for a mathematical study of a phenomenon. Like in the situations of “chase and prey” and of "vertical movement in an amusement park ride", students' understanding of geometrical optimization benefited of their everyday bodily experience in the sense of Núñez, Edwards and Matos (1999). In “chase and prey”, human experience of concrete situations helps to anticipate a trajectory taking into account the distance made by the prey during the pursuer's course. In "closeness on a curve", it is a common experience that a mobile on a trajectory comes closer and closer a given point outside this trajectory, up to a certain position, and then goes farther and farther. In the amusement park the "smooth" and "abrupt" transitions can be related to students' bodily experience. This is less clear with the phenomenon of tension in the cable of the suspension bridge, since tensions and especially variations of tension are generally not in the everyday experience. That is why experimenting with dynamometers in the first phase, and the emphasis on tensions in the other phases[5], was important. In some sense, when modelling real word phenomena, students' bodily experience can never be directly taken for granted. It has to be developed explicitly via physical experiences and discourse throughout the modelling process. The progression of students' awareness of the structure of a bridge is evidence of this development.

In this example of the suspension bridge, we used the connected working spaces framework to implement an innovative "jigsaw" organization of the group work. The observation shows that student did not progress as much as expected in the second phase especially with tasks B, C and D. Techniques, especially in analytical geometry, although known for a couple of years, could not be activated in this complex situation, and correcting an algorithm was difficult. However, in the third phase students could consider the whole modelling process, make connections between working spaces, and "fill some gaps" of the second phase. Figure 4 summarizes the connections made by students. I interpret those in terms of genesis in the sense of cognitive processes involved in students' activity (Kuzniak, 2013). The connections between the algorithmic and the mathematical function spaces, and between the statics and the algorithmic spaces deal with vizualisation, while the connections between the algorithmic and the geometrical spaces, and between the statics and the mathematical function spaces deal with the use of symbols as artefacts. Figure 4 shows also missing connections to the geometrical working space: the students did not prove the formula for the slope of a segment from the physical

[5] A study of the teacher's and the observer's intervention during the group work could give evidence of this. An experience like in the picture at http://goldengate.org/exhibits/images/GGB-exhibit3-2_1.jpg could also be useful.

consideration of the tension, and did not consider the gradient of the mathematical function as a limit of this slope. These missing connections deal with the discursive process of argumentation and proofs.

Finally the four working spaces and the "jigsaw" organization in a modeling activity, produce an answer to the question page 5 of how to make students understand interactions between concepts taught at upper secondary level, and the interactions between these concepts and other sciences or real world situations: at least at the informal level students get awareness of physical entities and of mathematical tools to work on these entities. It is the role of the collective synthesis in phase 4, not analyzed here, to bring this to a more formal level[6]. About the conceptualizations in this activity, I noted cognitive processes in visualization and use of symbols, and deficiency in the discursive process of argumentation and proofs. This finding is consistent with the limited possibilities for students to carry out proofs in actual secondary calculus[7]. It shows also that, in spite of this limitation, the work of modelling involves very rich cognitive processes in visualization and use of symbols, provided that suitable working spaces are organized. Another important observation is the central role of the algorithmic working space through connections to the three other spaces.

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[6] Two other situations involving the study of suspension bridges have been presented and analysed by Lagrange and al. (2015). This article especially analyses the phase of synthesis after a "jigsaw" group work.

[7] "Les différentes genèses (de l'analyse dans le secondaire) apparaissent comme étant déconnectées avec la difficulté de donner un sens à certaines reconstructions instrumentales et, surtout, à initier des preuves non iconique et discursive s'appuyant sur des propriétés clairement identifiées." (Kuzniak, Montoya ,Vandebrouck et Vivier, to appear)

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