# Teaching and learning about functions at upper secondary level: designing and experimenting the software environment Casyopée

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**Abstract:** Casyopée is an evolving project focusing on the development of both software and classroom situations to teach algebra and analysis at upper secondary level. In this paper, we sketch the rationales for the Casyopée project in relationship with the focus on functions in upper secondary curricula. To evaluate Casyopée's contribution, we present the design of an experimental teaching unit carried out in the ReMath European project focusing on the approach to functions via multiple representations for the 11$^{th}$ grade and some preliminary results.

**Keywords**: functions; software environment; Casyopée; multiple representations; instrumentation, theory of didactical situations**,** computer symbolic computation, algebraic learning, modelling, dynamic geometry.

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## Introduction

At upper secondary level, before university, students have to consolidate their algebraic proficiencies in order to prepare for calculus. In many countries, the choice generally done by curricula is to privilege functions. For instance according to the French curriculum students should learn:

> … to identify the independent variable and its set of values for a function defined by a curve, a table of data or a formula, to establish the value of the function for a given value of the variable in each register, to describe the behaviour of a function given by a curve, using a relevant vocabulary or a sketch[1].

The curriculum insists on the algebraic notation, and on the various equivalent expressions of a function:

> The notation f(x), already introduced before, and f will be systematically used… (Students should learn) to recognize various forms of an expression and to choose the most relevant form for a given work.

The idea of function has to derive from activities in varied mathematical and non-mathematical fields:

> Learning situations will come for instance from geometry, physics, actual life or historical problems. Students will have to reflect on language expressions like *a depends on b* in the common language and in mathematics.

More specifically, the curriculum points out problems related to geometrical dependencies as a basis for learning situations:

> It is possible to study geometrical situations, the independent variable being a length and the dependant variable an area. The problem is then often to look for a maximum, a minimum or simply a value.

The curriculum also encourages the use of technology:

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[1] Extracts of the French curriculum are the author's translation. The curriculum can be found at http://www.cndp.fr/secondaire/mathematiques/

> Computer tools can help a quasi-experimental approach to the fields of numbers and of geometrical objects. It favours students' more active attitude and commitment to the task. Possibilities for observing and manipulating are much wider. The opportunity of doing a great number of computations and to study as many cases as wanted helps to observe and verify properties.

The rationales and history of the Casyopée project at an earlier stage have been exposed by Lagrange (2005). The Casyopée team brings together teachers and researchers to take up the challenge of teaching about functions at upper secondary level, consistent with recent curricula. The team is concerned that while technology is able to offer multi-representational and symbolic manipulative capabilities very effective for solving problems and learning about functions, no tool really adapted presently exists for students' use. Dynamic Geometry software offers means for constructing operational figures and exploring co-variations and dependencies in these figures, but exploration is limited to numerical values. Students are neither encouraged nor helped to use algebraic notation and to work on algebraic models of geometrical dependencies. Computer Algebra Systems (CAS) exist to ease symbolic manipulation, but they are designed for more advanced users and it is difficult for secondary students to recognize functions and other objects as introduced by the curriculum. For instance, in most CAS, functions are considered over the whole set of real numbers without consideration of a set of definition. We also think that, in order to allow students deal with problems of geometrical dependencies, both in geometrical and algebraic settings, an environment should closely link a dynamic geometry component and an algebraic unit.

The question addressed by the Casyopée team deals with the possibility of developing a software environment fully consistent with the aims of the curriculum and that could help students to freely experiment, choosing their own way of solving and proving. Thanks to the Remath European project, the Casyopée team could progress toward this goal. This paper presents the Casyopée software and reports on an experiment with 11$^{th}$ grade students.

## Casyopée[2]

Casyopée has two main windows. The first one, (called the symbolic window) provides students with symbolic computing and representation capabilities as well as facilities for proving. The second one consists in a Dynamic Geometry (DG) window. Casyopée's two windows are closely linked, that is to say that objects in one window can be fully used in the other, and that the software provides the student specific aid to pass objects from one window to the other.

Real functions of one real variable are the central objects of Casyopée. A function is defined by a formula involving a function variable and a domain. As most other symbolic systems related to functions and numerical graphers define functions other the whole set of real numbers, without regard to the existence of formulas, this definition is a distinctive feature in Casyopée. It allows being consistent with the mathematical definition as well as providing realistic modelling: when designing a function as a model of a situation, often the function is not defined on the whole set of real numbers and often not on the whole set of existence of the formula. Casyopée provides means for creating sets of ordered real numbers, possibly including parameters, in order to define domains. These parameters can be treated both formally

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[2] A page for downloading Casyopée is available at http://casyopee.eu

and numerically by way of animation. Constraints can be set on parameters in order to adapt to all situations: for instance if the parameter is intended to model a measure, it can be defined as positive. Functions can depend on parameters. Expressions (that is to say formulas not involving a function variable but possibly involving parameters) can also be defined and treated. Thus Casyopée treats in a consistent way the algebraic objects generally included in upper secondary curricula about functions.

A wide range of construction capabilities is available within the DG window to build a figure including free points. Curves of functions can be drawn using the algebraic definition of functions (domain and formula). Because Casyopée is a DG system based on an underlying symbolic kernel (the free software Maxima), if offers constructions, like the intersection of a line and a curve, and the facility for exporting geometrical functions or expressions that are not provided by existing DG systems based on numerical calculations. Measures can be defined as "geometrical calculations" possibly including symbolic objects (parameters, functions, expressions…) created in the symbolic window. Casyopée can compute a domain and a formula for "geometrical" expressions or functions related to measures, providing a capability to express algebraically geometrical dependencies. This "export" capability that will be illustrated below is of a great help for students when modelling algebraically geometrical functional dependencies or expressions.

We recapitulate the main capabilities available in Casyopée's symbolic window:

- operations on expressions or functions (e.g. expanding or factoring formulas, integrating or differentiating functions, solving equations…);
- graphic representations of functions (with different functionalities such as zooming, changing axis scales…);
- numerical or formal data on functions (such as particular values or limits);
- proof capabilities (theorem are available inside Casyopée that user can apply to functions in order to prove signs or extrema or variations or zeros);

We recapitulate the main capabilities available in Casyopée's DG window:

- geometrical construction (allowing to build figures, including free points on objects such as segments, circles, straight lines…); Casyopée's algebraic objects –functions, expressions and parameters- can be used. For example, parameters and functions can be involved in geometrical objects definitions (eg a coordinate points defined by its two coordinates, expressed as numerical values or values including parameters);
- creation of geometrical calculations (well-formed formulas involving measures and symbolic objects);
- numerical explorations of measures;
- capabilities to determinate and link independent and dependant measures that will be used to define "geometrical" functions;
- "exportation" of these geometrical functions into the symbolic window of Casyopée;
- "exportation" of geometrical expressions of measure depending on no free point into the symbolic window of Casyopée.

**Solving a problem of functional dependency with Casyopee**

In order to explain the software's functionalities, we expose now the type of problem whose resolution can take advantage of Casyopée, and how. This is an example:

Consider a triangle ABC. Find a rectangle MNPQ with M on [oA], N on [AB], P on [BC], Q on [oC] and with the maximum area

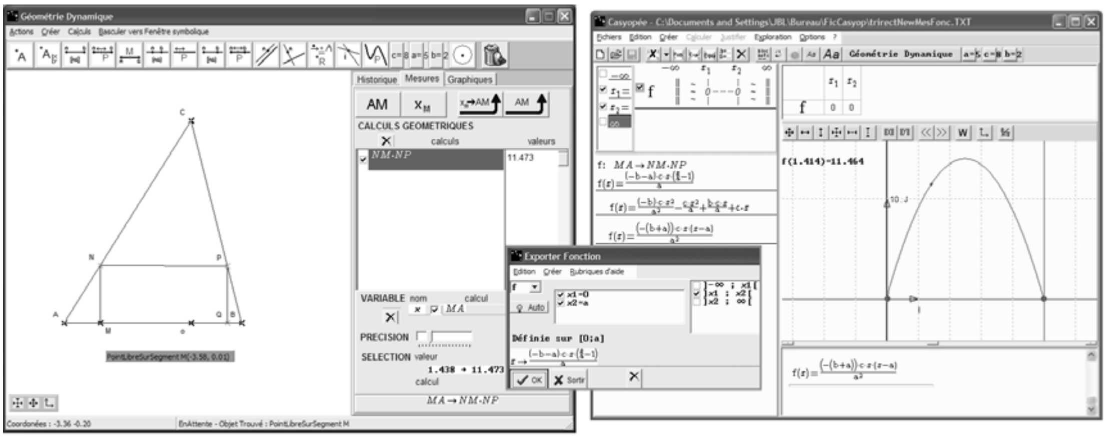


Figure 1: Casyopée's symbolic and DG windows and the exportation form.

Constructing a generic triangle ABC in the geometrical window can be done after creating parameters in the symbolic window. For instance, the points can be *A(-a;0), B(0;b)* and *C(c;0)*, a, b and c being three parameters. Then one can create a free point M on the segment [oA] (o being the origin) and the rectangle can be constructed using dynamic geometry capabilities.

In the Geometric Calculation tab (Fig.1) one can create a calculation for the area of the rectangle MNPQ and then define an independent variable. Numerical values of calculations and of the variable are displayed dynamically when the user moves free points. The user can then explore the co-dependency between these values. If this co-dependency is functional (i.e., the calculation depends properly on the variable) it can be exported into the symbolic window and Casyopée automatically computes the domain and the algebraic expression of the resulting function. Otherwise, Casyopée gives adequate feedback.

After exporting into the symbolic window, one can work on various algebraic expressions of the function and on graphs. For instance, one can use properties of parabolas, or algebraic transformations or Casyopée's functionality of symbolic derivation to find the answer to the question. One can also use the graph of the function to conjecture about the area maximum.

## Theoretical frameworks used to build and analyse the experiment

To evaluate the support that Casyopée can bring to the learning of functions, we built an experimental teaching unit at $11^{th}$ grade. We present first the frameworks that helped us to build this experiment and to interpret our observations. Then we present the experiment and we report on the observation of the last session where students used the wider range of representations.

The first framework is based upon the notion of "setting" introduced by Douady (1986). According to Douady, a *setting* is constituted of objects from a branch of mathematics, of relationship between these objects, their various expressions and the mental images associated with. When students solve a problem, they can consider this problem in different settings. Switching from a *setting to another* is important in order that students progress and that their conceptions evolve. Students can operate these *changes of setting* spontaneously or they can be helped by the teacher. The setting distinguished here are geometry and algebra,

We also rely upon the notion of *registers of representations* from Duval (1993). Duval stresses that a mathematical object is generally perceived and treated in several registers of representation. He distinguishes two types of transformations of semiotic representations: *treatments* and *conversions*. A treatment is an internal transformation inside a register. A conversion is a transformation of representation that consists of changing of a register of representation, without changing the objects being denoted. It is important that students recognize the same mathematical objects in different registers and they get able to perform both treatments and conversions.

Here we distinguish the geometric and the algebraic settings corresponding to Casyopée's two main windows. In these two settings, the functions modelling a dependency are different objects: a relationship between geometric objects or measures in the geometric setting, and an algebraic form involving a domain and an expression in the algebraic window. In the above problem, students have to switch from the geometric to the algebraic settings and back, to be able to use symbolic means for solving questions that were formulated in the geometric setting. As explained by Lagrange & Chiappini (2007), we expect that, working in the geometric setting, students would understand the problem and the objects involved, and that after switching to algebra, this understanding would help them to make sense of the objects and treatments in the algebraic setting.

Inside each of these two settings the functions can be expressed in several registers. In geometry, especially with dynamic geometry, functions can be represented and explored in different registers: covariations between points and measures, or between measures, or functional dependency between measures. In algebra, functions can be expressed and treated symbolically, by their expressions, by way of graphs and of numerical tables. Mastering these expressions and treatments, and flexibly changing of register are important for students' ability to handle functions and acquire knowledge about this notion.

A third framework is the *instrumental approach*, based on the distinction between artefact and instrument. An actefact is a product of human activity, designed for specific activities. For a given individual, the artefact doest not have an instrumental value in itself. It becomes an instrument through a process, called *instrumental genesis*, involving the construction of personal schemes or the appropriation of social pre-existing schemes. Thus, an instrument consists of a part of an artefact and of some psychological components. The instrumental genesis is a complex process; it requires time and depends on characteristics of artefacts (potentialities and constraints) and on the activities of the subject (Vérillon & Rabardel, 1995).

In the case of an instrument to do or learn mathematics like Casyopée, the instrumental genesis involves interwoven knowledge in mathematics and about the artefact's functionalities. Artigue (2002) showed how this genesis can be complex, even in the case of simple task like framing a function in the graph window. More generally, the many powerful functionalities of CAS tools have a counterpart in the complexity of the associated instrumental genesis (Guin & Trouche, 1999). We are then aware that we must take care of students' genesis when bringing Casyopée into a classroom. Moreover, Casyopée offers a multiplicity of representations in two settings and in several registers. Understanding and handling these representations involves varied mathematical knowledge. Students have then to be progressively introduced to these representations, taking into account the development of their mathematical knowledge.

Constructing the sessions of the experiment, we also used the Theory of Didactical Situations (Brousseau, 1997) as basis for designing tasks. According to this theory, learning happens by means of a continuous interaction between a subject and a milieu in an *a-didactical situation*. Each action of the subject in milieu is followed by a retroaction (feedback) of the milieu itself, and learning happens through an adaptation of the subject to the milieu. Thus, with regard to Casyopée use, learning does not depend only on the representational capabilities of this software, but also on tasks and on the way they are framed by the teacher. Within this perspective, we looked for situations in which students interact with Casyopée and receive relevant feedbacks. For example, to solve the above problem, students have to choose between different independent variables to explore functional dependencies in the geometrical window and to export a dependency into the algebraic window. In case the variable is inadequate, the feedback they receive is a message from Casyopée. In other cases, the algebraic expression automatically produced by Casyopée can be more or less complex, which is another feedback: too complex expressions have to be avoided in order to ease the subsequent algebraic work.

Concerning the methodology, we use *didactical engineering* (Artigue, 1989), a method in didactic of mathematics, to organize and evaluate the experimental teaching unit, and to answer the research questions. The treatments and interpretations of collected data based on an internal validation which consists in confronting *a priori* analysis of the situation with *a posteriori* analysis. This method produces an ensemble of structured teaching situations in which conditions for provoking students' learning have been planned.

## The experiment

Our experimental teaching unit consisted of six sessions. It was experimented in two French 11$^{th}$ grade classes. It was organized in three parts. Consistent with our sensitivity to students' instrumental genesis, each part was designed in order that students learn about mathematical notions while getting acquainted with Casyopée's associated capabilities:

The first part (3 sessions) focused on capabilities of Casyopée's symbolic window and on quadratic functions. The aim was that students became familiar with parameter manipulation to investigate algebraic representations of family of functions, while understanding that a quadratic function can have several expressions and the meaning of coefficients in these expressions. The central task was a "target function game": finding the expression of a given form for an unknown function by animating parameters.

The second part (two sessions) aimed first to consolidate students' knowledge on geometrical situations and to introduce them to the geometrical window's capabilities. The central task was to build geometric calculations to express areas and to choose relevant independent variables to express dependencies between a free point and the areas. It aimed also to introduce student to coordinating representations in both algebraic and geometrical settings, by way of problems involving areas that could be solved by exporting a function and solving an equation in the symbolic window.

Finally, in the third part (one session) of the experimental unit, students had to take advantage of all features of Casyopée and to activate all their algebraic knowledge for solving the optimization problem presented above.

Below, we give some insight on how we are currently exploiting this experiment with regard to our question about Casyopée' potential for learning about functions. We limit ourselves to the final session for which the problem and the students' instrumental genesis should allow to take full advantage of this potential. We draw some elements of a priori analysis of this session and we compare with the a posteriori analysis of students' achivements.

## The situation in the final session: elements of *a priori* analysis

### *Tasks*

The problem was presented by the teacher by animating a figure in Casyopée's geometrical window:

*Let a, b and c be three positive parameters. We consider the points A(-a;0), B(0;b) and C(c;0). We construct the rectangle MNPQ with M on [oA], N on [AB], P on [BC] and Q on [oC]. Can we build a rectangle MNPQ with the maximum area?*

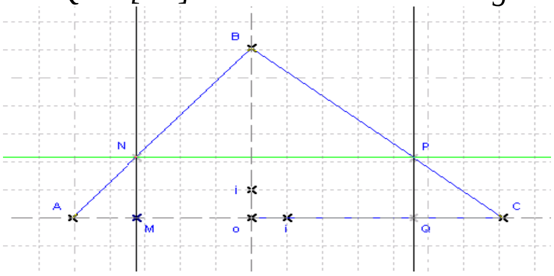


Fig. 3: The figure built in Casyopée

The tasks proposed to students were then:

- The construction of the rectangle MNPQ: students are required to load a Casyopée file with the parameters' definition and the triangle, then to complete the figure by building the segments [oA], [AB], [BC] and [oC] and to create the free point M and the rectangle's vertexes.
- To create a geometrical calculation for the area of the rectangle MNPQ: this can be obtained by the product of the lengths of two adjacent sides, e.g. MNxMQ
- To explore the situation by moving the point M on the segment [oA].
- To prove the conjecture by algebraic means.

The teacher also asked students to write the proof, indicating their choice of variable and using results displayed by Casyopée. Finally, students were expected to visualize the answer in the geometrical window.

### *Covariations and representation of functional dependencies*

This situation involves two settings and different registers. Students can conjecture the answer to the question by exploring numerical values of the area in the geometrical setting. They can explore the variation of the area in different ways corresponding to different registers of representation. First, they can observe co variation between the point M and the area, looking at the values of the calculation they created for the area of the rectangle, noting that when M moves from A to B the value grows then decreases, with a maximum value when M is the middle of [oA]. They can also observe co variation between a measure involving the free point M and the area. For instance, they can observe together the values of the distance oM and of the area. Finally, they can choose an independent variable involving M and observe the functional dependency between this variable and the area.

In the algebraic setting students can apply different algebraic techniques to the algebraic form of the function in order to find a proof. Exporting a function with Casyopée, one obtains a more or les complex algebraic expression reflecting the calculation's structure. Students then need to expand this expression to recognize a quadratic function. They can then apply their knowledge about these functions to prove the maximum. It is possibly not easy for them, because of the three parameter involved.

They can also use the graphical representation in this algebraic setting to explore the curve, complementing the exploration they did in the geometrical setting: the parabola is familiar to the students and they can easily recognize a maximum.

The situation is partly a-didactical. In each setting, students interact freely with Casyopée and use the feedbacks to understand the situation. Nevertheless, some key points like passing from a co variation to a functional dependency are expected to be difficult for students, although the corresponding action (choosing an independent variable) has been presented in the preceding sessions. Passing from one setting to the other is expected to be far from obvious for students. The corresponding actions in Casyopée (exporting a function in the symbolic window, interpreting a symbolic value in terms of position of a point) have also been presented before, but it is the first time that students have to do it by themselves.

Students can choose their own independent variable between possible choices (oM, $x_M$, MN, MQ…) with consequences upon the algebraic expression of function. They can do it alone but it is expected that the teacher mediation will be necessary. It is also possible that they will want to change their choice of a variable in order to obtain a simpler algebraic expression of the function.

We expect a great variety of uses of representations reflecting students' free interactions with the situation. Some students can stay a long time exploring co variations and need teacher mediation to go to functional dependency while others pass more or less quickly to the algebraic setting to consider the function. In this setting, some can prefer to explore graphs, while others prefer working on algebraic expressions. It is possible that some students find too difficult to apply algebraic techniques to the general expression (i.e. with parameters) and prefer to work by replacing these parameters by numbers. In any case, we expect that students will consider several representations, make sense of them and make links between them.

## Elements of a posteriori analysis

During the experiment, we observed selected teams of students. In this paper, we first focus on a team of two students, which according to the observation in the first five sessions had a favorable instrumental genesis and then we report on the other groups.

### *The explorations in different settings and registers*

Creating a geometrical calculation for the area of the rectangle, they typed MNxMP instead of MNxMQ by mistake. They moved M and observed growing numerical values of this calculation, while, for some positions of M the area was visibly decreasing. This first feedback allowed them to correct the geometrical calculation.
Like most students they had difficulties in choosing an appropriate independent variable, confusing the independent variable and the calculation. They needed help from the teacher to activate the correct button. They chose at first NP. They moved for a long time the point M and observed how numerical values of this variable and of the area MN×MQ changed. They found an optimal value and interpreted it: "(the optimum) is when N is the midpoint of [AB] I believe, and P is the midpoint of [BC]". The teacher asked them for a proof. A student suggested an equation in an interrogative tone. Actually, the problems solved in sessions 4 and 5 were about equalities of areas and have been solved by way of an equation.

The teacher guided them to export the function, but they found the resulting expression too complicated. Then they choose another independent variable MQ, and got the same expression after exporting again the function. Finally, they chose $x_M$ as an independent variable, obtained the algebraic expression b(x-1/ax)(a+c-a(x-1/ax)-c(x-1/ax)) and expanded it into a quadratic polynomial.

### *Proving the maximum*

The team graphed the function, recognized a parabola, and said that they do not know how to determine the maximum's x-coordinate. Then they wanted to apply an algebraic formula to get this x-coordinate and used Casyopée to expand the expression. For some reason they got a non parametric expanded expression, the parameters being instantiated. Then it was easy for them to obtain by paper/pencil a numerical value of the maximum's x-coordinate. Then they returned to the geometrical window, checked this result and generalized, saying that the maximum is for $x_M$=a/2. They did not attempt to prove this generalized property by working on the parametric expression and then they only partially solved the problem.

### *Other groups*

The other students' activity was very diverse with regard to the time they devoted to each register and the difficulties they had to go from one to another. Some stayed a long time in geometrical exploration. A minority of these students did not really work on mathematical functions, except for graphs. No student went directly to computing symbolically an optimal value after exporting a mathematical function: they first explored the graph. A few students however went more quickly towards this, some doing no or very little exploration. This diversity of activity is for us an indication that students could work with Casyopee at their own pace, developing activities that they could understand.

The observation reported above is then globally consistent with the a priori analysis. The students used more or less all registers of representation. The independent variable was recognized as the central feature of the solution, allowing connections between registers. Casyopée offered means for exploration and various feedbacks that helped this recognition. The students' instrumental genesis helped them globally to interact with Casyopée. Nevertheless, important actions like choosing a variable and exporting a function were still unfamiliar. Also understanding an optimization problem was not easy and students were influenced by the problems of equalities of areas they solved before. Although they used parameters before and they understood the generalized problem, using parametric algebraic expressions was also difficult. These difficulties are central in the idea of functions. Students dealt with theses difficulties, Casyopée acting as a milieu providing adequate feedbacks, but also with the teacher's help.

## Perspectives

Our perspectives are about the teacher's activity when using Casyopée. In this experiment we worked with very experienced teachers, members of the Casyopée team. We observed them help students in crucial episodes, like changing settings and we want to examine from our observations if and how this help contributed to students' learning, beyond the solution of the problem. We are also much interested on the phases of collective discussion moderated by the teacher. This work is initiated by Maracci and al. (to appear) by comparing the experiment presented in this paper with another experiment in Italy driven by the semiotic mediation theoretical framework that gives much importance to class discussion managed by the teacher.

Disseminating Casyopée and Remath results among teachers in connection with the other Remath work packages is also an important perspective. We selected a region of France (Brittany) where local authorities are specially interested by Casyopée's features. A group of teachers has been set up with the help of the French INRP (National Institute for Pedagogical Research) and the IREM (Institute for Research in Mathematics Teaching) of Rennes. Although already interested by digital technology, these teachers were not involved before in Casyopée's design and experimentation. We expect them to bring a fresh vision of Casyopée's features and to prepare scenarios of use that will be disseminated to other mathematics teachers together with Casyopée on the professional digital workspace for teachers in Brittany. We plan to set up a regional community of users, and later to extend to wider communities.

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