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Technology and Mathematics Education: A Multidimensional Study of the Evolution of Research and Innovation

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ABSTRACT

This chapter will highlight the interest and necessity of considering a plurality of perspectives (or dimensions) when addressing the issue of the integration of information and communication technologies (ICT) into the teaching and learning of mathematics. It will also show how this multidimensional perspective can be efficient for an analysis of the existing literature.

The paper draws on a meta-study of a comprehensive corpus of publications about research and innovation in the world-wide field of the integration of ICT from 1994 to 1998. For this study we built a multidimensional framework and a data analysis procedure, and obtained a synthesis of literature. The study of ten research papers that the statistical procedure made appear as paradigmatic examples helped to discern an evolution towards more awareness of the complexity of ICT integration. The multidimensional framework aims to provide innovators and researchers with a set of references to deal with this complexity.

INTRODUCTION

ICT in Mathematics Education is a very active field of research and innovation. On the one hand, there is a great amount of literature offering a wide range of theories, methodologies and interpretations, and generally stressing the potentialities for teaching and learning. On the other hand, integration into school institutions progresses very slowly compared with what could be expected from

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the literature. For us, this discrepancy is evidence of a lack of precise understanding of what really happens when introducing ICT into schools. Teaching and learning involve complex processes, and bringing in ICT adds even more complexity. Our hypothesis is that literature evolves in tackling more and more aspects of this complexity, but lacks global and accurate understanding. Thus we think that meta-studies of the literature are necessary.

After a call for research issued by the French Ministry of Education, which was concerned about the poor returns from research and innovation, we started a study of a comprehensive corpus of research and innovation publications in the world-wide field of ICT integration. We chose the period from 1994 to 1998 which appeared particularly worthy of a study, because during these years the classroom use of technology became more practical, and literature matured, often breaking with initial naive approaches. We did not focus attention on findings but considered more general characteristics, and this will be explained below.

Doing the study we built a framework, of several dimensions, in order to account for trends in the corpus. This paper will develop two outcomes. The first is a synthesis of the literature over this key period, using this framework and a statistical procedure. The second is the framework itself. Our intention is that, together with the synthesis, it could help in grasping the evolution of the literature and provide research and innovation with a wider understanding of ICT integration.

The first section of this chapter will document how the multidimensional framework was built and the global synthesis obtained with it. Then we will focus on a selection of ten research papers that a statistical procedure made appear as paradigmatic examples. Each dimension will be analysed with respect to their argumentation. A concluding section will present our interpretation of the evolution of innovation and research.

THE DEFINITION OF DIMENSIONS

Methodology

We were aware of the methodological interest in a meta-study because of limitations of informal methods as emphasised by Bangert-Drowns and Rudner (1991): "The amount of information on a given topic (...) is often overwhelming and not amenable to summary. (...) it is difficult to determine if outcome differences are attributable to chance, to methodological inadequacies, or to systematic differences in study characteristics. Informal methods of narrative review permit biases to remain easily undetected" (p. 1).

We also wanted to go beyond a synthesis of outcome or findings. The reason is that syntheses of findings generally conclude by favouring strong potentialities of ICT but give few explanations for the contrasting poor real classroom use.

For instance, in a synthesis of studies on the use of technology to learn algebra, Haimes and Malone (2001) state that: "(technology) can be incorporated into

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existing practices and curricula in order to facilitate student learning and conceptual development" and, observing the poor real actualisation of these potentialities, assume that "the pace of development of technology may have exceeded that of professional development of teachers" (p. 285).

The factors that authors can really control in their investigations are necessarily limited with regard to the above mentioned complexity and no comprehensive framework is currently available that could help to understand the effects of this limitation. Thus, whilst authors can draw conclusions about the strong potentialities of technology, they can be unaware of obstacles to their actualisation. These obstacles can exist even when circumstances like available resources, strong official support and substantial professional training should favour the integration.

Thus, as a difference with classical meta-studies, we did not focus just on the findings of publications but considered also characteristics like the questions addressed, approaches, cognitive theoretical background, etc. We expected that analysing this material would help to identify as many as possible aspects of the complexity of the integration, some of them widely addressed and others less considered by the literature.

We did that in two stages. In the first stage, we considered a corpus as large as possible. It helped us to have a broad look at research and innovation, and to organise the above aspects into 'dimensions of analysis'. In the second stage, we selected a sub-corpus and did a statistical analysis within each of the dimensions.

In the first stage, we used a variety of international sources (the 'Zentralblatt für Didaktik der Mathematik' database with the entry 'Computer Assisted Instruction', four international journals on mathematical education, seven international journals on computers for mathematics learning, books with chapters on technology and mathematics education, etc.) as well as French works (professional and research journals, dissertations, research and official reports, etc.) This resulted in a corpus of 662 published works.

A selection of papers for the second stage analysis was necessary because not all papers in the first stage corpus had sufficient matter and also because working on each paper was expected to take several hours. We still decided to have a large enough selection to respect the diversity of approaches and to avoid biases.

We selected 79 papers, judging this selection to be a good compromise between in depth analysis and respect of diversity. For each of these papers, one of the participants of the project established a detailed review showing the above mentioned study characteristics: theoretical background, the details of the questions addressed, methodology used, specific findings and an appreciation by the reviewer.[1]

The exploitation of this 'second stage' corpus was achieved through a statistical classification giving informative clusters of publications with regard to the various dimensions.

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The First Stage Analysis

We started this study with the hypothesis that the great variety of types of publications in the field of the use of ICT gives evidence of a lively field but also makes difficult the search for convergent findings. We had also an intuition of what this variety would be and the simple quantitative treatments that we did first were intended to provide evidence of the diversity of the publications as well as to check the conformity of the corpus compared with what we thought of the general production.

QUANTITATIVE TREATMENTS

Tables 1 to 4 give the repartition of the first corpus with regard to general indicators.

Table 1 shows that research publications were not a majority. The literature about ICT includes classroom innovation and pure speculation as well as research studies. Table 2 shows that few papers were about arithmetic and algebra alone, probably reflecting the smaller amount of writings about the pre-college level, outside geometry, in the corpus. A number of papers did not specify a mathematical field, focusing on the support of technology in 'general' mathematical learning.

Various technologies were represented (Table 3). Although simple calculators are everyday instruments for many students all over the world, papers tended to focus on 'smarter' technologies, like computer software or symbolic calculators. Emergent technologies (Internet, etc.) although very present in discourses were rarely addressed by papers in our corpus.

There was also a great variety of countries (Table 4). The bias towards France is because it was easier to get hold of publications in France like doctoral dissertations, professional teacher journals, congresses and professional meetings reports that we wanted to include besides papers from international journals.

QUALITATIVE CLASSIFICATION OF 'PROBLÉMATIQUES'

After these quantitative treatments, we did a qualitative analysis to have a general idea of the diversity of the approaches of the publications in the initial

Table 1. Type of analysis

Presentation of product	20%
Experimentation, Innovation	20%
Research report	37%
General reflection	15%
Not specified	6%

Table 2. Mathematical field

Geometry	23%
Calculus (or Algebra and Calculus)	14%
Algebra	11%
Graph/functions	10%
Arithmetic	5%
Other fields	7%
No specific field	30%

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Table 3. Type of technology

Calculators	12%	Numeric and scientific	2%
		Graphical	4%
		Symbolic	6%
Computer software	50%	(Dynamic) geometry	16%
		Other microworlds	8%
		Spreadsheet	6%
		Symbolic and graphing systems	21%
Other technology	21%	Tutorials	2%
		Intelligent environments	4%
		Internet	2%
		Multimedia	3%
		Hypermedia	3%
Not specified	17%		

Table 4. Countries

France	29%	Israel	2%
United States	23%	Australia	2%
United Kingdom	11%	Eight papers or less (1%):	
Germany	6%	Finland, Japan, Netherlands, Spain, Switzerland, Portugal, China,	
Italy	3%	Brazil, Canada, Denmark, Norway, Mexico, New Guinea, Belgium,	
Austria	2%	Greece, Ireland, Russia, Sweden	
		Not specified	12%

corpus before selecting the second stage corpus. We focused our attention on the ‘problématiques’. We explained above why it is important in our research to study the questions addressed in the field of ICT as material for building a comprehensive interpretative framework. We call here a “problématique of a publication” the set of connected problems or questions that the publication addresses.[2]

For each publication in the first stage corpus, we established a small text summarising this ‘problématique’. An analysis of the ‘problématiques’ of the whole corpus would have been too time-consuming. So we decided to restrict this analysis to a specific field: the educational use of Computer Algebra Systems (CAS). We chose this field because it appeared with a high percentage (27%),[3] reflecting the great number of papers written in this field. Among other IC Technologies, symbolic computation raises much interest in mathematics education, even when not so much classroom use is observed, and so it is especially relevant in relation to our concern for the appropriateness of questions posed in the papers.

Another reason was that the corpus included all the papers issued from a

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journal dedicated to CAS educational use, the IJCAME.[4] As this journal encourages research papers as well as articles about teaching issues, and activities for classroom use and opinions, we were particularly sure of having a variety of papers in this field. The CAS sub-corpus also included thirty percent of publications on CAS use from a great variety of other journals or books, and we avoided a biased panel that could result in the analysis of papers from a single journal.

We found that the publications of the CAS sub-corpus could be classified into five 'types of problématique' that we summarise below. In this qualitative approach, papers have been classified from a general impression of the reader, rather than from objective criteria and so the percentages given for each type are only indicative of their relative importance.

Technical descriptions (53%). These papers stress the capabilities of CAS that they find relevant for educational use. They generally emphasise the educational potentialities of ICT like visualising, modelling and programming that should make students' use beneficial. Optimism is a very common feature of these papers and only a few articles stress the need for students' training to avoid pitfalls.

Innovative classroom activities (9%). These papers report actual CAS use in classrooms. Presentations of national projects for experimenting and integrating CAS (like French or Austrian projects) and of new curricular projects influenced by technology are examples of these. When authors give reasons why CAS might be beneficial in these activities, we find an emphasis on a better conceptualisation resulting from the above potentialities but also on the time saved on the 'boring' procedural execution of algorithms. Technology is also offered as a solution to learning problems. Sometimes data are given from the authors' experience of teaching, but we rarely found a real discussion.

Assumptions about improvement (12%). These papers are more research-oriented studies, with hypotheses, experimentation and conclusions. Hypotheses come from general views on mathematics teaching and learning, and on technology, and are often backed by a cognitive theory. They assume improved understanding and better problem solving abilities as a result of ICT use by students. An experiment is then created to bring these assumptions into operation. It is based on an innovative educational use of symbolic computation, generally a curriculum modified by the introduction of CAS. The main question is the efficiency of this experiment to get the assumed improvements. The author generally provides an 'external' comparison of 'experimental' and 'control' students. It aims to provide evidence of this efficiency rather than data for a discussion about the innovation.

Questions about the use of technology (21%). In contrast with the above papers these do not presuppose any advantages of this use. They present innovations, experimentation or examples of use not for themselves, but as a tool to address the questions. There is a great variety of questions. General and theoretical questions consider the limits and constraints of CAS, the tasks, procedures and types of understanding promoted by technology, the design of software, tutorials

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and teacher training. Many questions focus on situations of use of CAS like the co-existence of students who used DERIVE at home and other students who did not, the comparison of procedures using various software for a given task, the role of CAS in exams, etc. Questions about the students start from their attitudes toward software, from difficulties associated with the use of technology or in relating mathematical ideas to CAS output or in exploiting the knowledge developed through this use to help further learning.

Integration (5%). These papers address the issue of the conditions for CAS to be used such as paper and pencil in the everyday practice of teaching and learning in existing school institutions. These conditions may concern the type of knowledge to be taught, as well as the solving strategies or procedures in usual or not too modified problems. Briefly speaking the 'ecological' conditions for the use of CAS are investigated. Like the preceding papers, these publications are centred on questions and share their methodology with these. We separated them because we thought that they open distinctive fields of questions.

The Dimensions

The above classification shows a variety of approaches in the CAS field, reflecting the diversity of reflection, research and experiment on the use of ICT. The papers with a mere technical approach of possible use of ICT (type 1) prevail (more than one in two). The mass of papers produced in this approach is representative of the interest raised by a technology like CAS among educators. Papers arguing in favour of classroom innovations (type 2) are much less common, but seem useful especially when they report on long-term experiments of students' use of CAS. A weakness is that they generally do not discuss the teacher's options, but nevertheless they could advantageously be a basis for specific studies of this dimension.

Most of the papers of type 1 and 2 lack sufficient data and analysis, and we could not integrate them into the second stage corpus. We had nevertheless to remember their great number. Obviously, one cannot really rely on their conclusions for the use of ICT in the classroom, but we stress them as an effective basis to think of situations of use.

Among research papers more adequate for a second stage analysis, our classification distinguishes publications starting with assumptions on expected improvements resulting from students' use of CAS (type 3) from others starting with questions about this use (type 4 and 5). There seem to be two different ways of doing research that a further analysis could investigate. That is why we will consider as a first dimension of our analysis the general approach of each paper about ICT in education ('problematiques', hypotheses, methodology, validation means, etc.).

The assumptions and questions have in common their focus on the epistemological and semiotic dimensions: they generally consider the mathematical knowledge at stake in technological settings and the possible effects of its computer implementation, and they often regard the paper/pencil and the technological

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registers of expression and their connections. These issues make a second dimension, the 'epistemological and semiotic' one.

Type 3 to 5 papers share also a common interest in a 'cognitive' approach: the assumptions of improvements and the question about the use of ICT are generally based on a theoretical background about the student's functioning and learning processes. We observed a great diversity. A more or less precise 'constructivist' general approach is expressed in a number of different concepts and theories. For instance, in this CAS sub-corpus, we found references to elaboration like the 'APOS theory' (Dubinsky, 1991), the notion of 'procept' (Tall & Thomas, 1991), of 'amplification and reorganisation' (Pea & Roy, 1987), of 'constructionism' (Harel & Papert, 1991), and even to general notions like 'scaffolding' (Wood, Bruner & Ross, 1976). In this diversity, few convergence appears and only at a general level. One of these is a greater sensitivity to issues like visualisation, connection of representations and situated knowledge. This cognitive approach will be a third dimension.

The last class (type 5) gathers a minority of papers explicitly addressing the issue of integration, which implies a study of questions like those in type 4 papers, but with a specific approach of ecologically sustainable use. Questions about tasks, procedures and conceptualisation, as well as about CAS as an instrument exist in the type 4 papers. Type 5 publications investigate these issues more systematically and helped us to organise them into the two dimensions that we briefly characterise below:

- The 'instrumental' approach of ICT distinguishes a technological artefact and the instrument that a human being is able to build from this artefact. While the artefact refers to the objective tool, the instrument refers to a mental construction of the tool by the user. The instrument is not given with the artefact, it is built in a complex instrumental genesis and it shapes the mathematical activity and thinking (see Trouche, 2000).
- The 'institutional' approach investigates to what extent content to be taught as well as tasks and 'techniques' (ways of doing tasks) are affected by the institution in which they are taught. Institution has to be understood in a broad sense: a specific classroom with a specific teacher may be considered as an institution as well as a general school level for a country. In this approach, techniques are considered to participate in the conceptualisation rather than as routine manipulation. The interest of ICT in education is in the new techniques they offer and in the interaction of these techniques with the 'old' paper/pencil techniques (see Lagrange, 2000).

A Multi-Dimensional Framework

The dimensions for the analysis of the integration of technology were derived from the above qualitative analysis. Five dimensions briefly described above were:

1. the general approach of ICT in education,

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2. the epistemological and semiotic dimension,
3. the cognitive dimension,
4. the institutional dimension,
5. the instrumental dimension.

We considered two more dimensions, not very visible from the above classification, but which we find important.

1. the 'situational' dimension which deals with the influence of ICT on learning situations,
2. the 'teacher' dimension which looks at the teacher's beliefs and at the way (s)he organises the classroom activity.

For each of the dimensions a set of indicators was designed resulting into a grid. Table 5 outlines the indicators for each dimension.

Using this table, we looked for trends in the second corpus using a statistical procedure based on a cluster analysis. Applying this procedure to indicators in each dimension, we obtained clusters of papers sharing specific indicators. This

Table 5. Indicators of dimensions

1. General approach of the integration	• Type of hypothesis (assumption of improvements, questions, etc.) • Methodology and validation processes (comparing experimental and control groups, comparing a priori analysis and expectations with an experiment, etc.)
2. The epistemological and semiotic dimension	Influence of ICT • on the mathematical knowledge and practices • on the representatives used in this activity
3. The cognitive dimension	• Cognitive frame (constructivist, socio-cultural, …) Concepts used (schemes, webbing, etc.) • Cognitive role of ICT (visualisation, expression, connection, etc.)
4. The institutional dimension	• Interaction of ICT with tasks and techniques in the culture of a school institution • Role of instrumented techniques in conceptualisation of mathematics
5. The instrumental dimension	• The tool's possibilities and constraints • Instrumentation processes
6. The situational dimension	Influence of ICT on • the structure of the situation • students' solving strategies • the didactical contract
7. The teacher dimension	• Teacher's beliefs and representations of mathematics and of ICT • New teaching situations • Influence of research and pre/in service programs

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procedure gave data to explain each cluster by showing the indicators of the table which were preferentially shared by the publications of the cluster. The procedure also selected one or two papers at the centre of each cluster, which were the publications statistically best representative of the cluster.

Innovation and Research in the Years 1994–1998

Regarding the type of hypothesis (dimension 1), the statistical analysis separated two clusters. A large cluster was centred on questions while a smaller started from assumptions of improvements. Both shared nevertheless a common focus on the interaction between students and technology, questioning very little the other factors of teaching and learning (teacher, institutions, etc.). Epistemological and semiotic considerations (dimension 2) also separated into two clusters. One of them gathered papers in which the analysis was more precise than in the other (we will specify this below). We took this as evidence of a corpus providing a mass of data and results about the learner and the attention paid in this corpus to the relationship between the content knowledge at stake and the new teaching means and representations provided by ICT.

The significant clusters in the other dimensions were smaller and their interpretation is less obvious.

In the cognitive dimension, a first cluster was representative of a constructivist approach. The second focused more on visualisation and the third on the contextual character of the meaning that students can develop when using ICT. The classification in this dimension was not very consistent and confirmed our expectation of a dispersion of the theoretical backgrounds.

References to the ‘instrumental’, ‘institutional’, and ‘situational’ dimensions were sparse and there is no easy interpretation of the clusters. Very few papers considered the teacher dimension and so we did not obtain a convincing classification of papers in this last dimension.

The general picture of ICT in the teaching and learning of mathematics emerging from this analysis is thus of a field where publications about innovative use or new tools and applications dominated. Research studies differed with regard to the way they considered the potentialities of ICT, but they converged in a focus on the student and in an emphasis on epistemological and semiotic aspects. This is certainly an interesting feature, that the approaches of ICT in mathematics pays attention to the learner and considers the knowledge at stake as an important component. The wealth of cognitive elaboration that we found can be considered as a consequence of the data and results that the emphasis on the student and the knowledge provided.

On the other hand the evidence appears to show that this emphasis is not sufficient in itself when little is known of instrumental constraints, ecological viability, of the new economy of didactical situations and of the influence of the teacher.

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AN ANALYSIS OF PARADIGMATIC EXAMPLES

We now focus on ten papers situated by the statistical procedure near the centre of the informative clusters that we consider as possible paradigmatic examples. The data will be a summary of each paper's argumentation (approach, study, results) appearing in the annex. This argumentation will be discussed with respect to the other papers belonging to the same cluster and thus considering the same dimension with a similar approach.

In the statistical analysis, the dimensions 1 and 2 appeared very consistent towards concerns related to the analysis of the knowledge and the associated representations, together with the way students interact with technology. Thus from now, we will aggregate these dimensions. The teacher dimension appeared to be rarely comprehensively addressed by research and innovation and thus we will not consider below this dimension alone. We will analyse it inside the institutional dimension, by way of a paper considering both the teacher and the institutional concern for the viability of ICT in education.

General and Semio-Epistemological Approach

In this aggregation of the first two dimensions, we consider two papers (Tall, 1993; Kieran et al., 1996). Our statistical procedure shows them in separate clusters with regard both to dimensions 1 and 2. As three years passed between their respective publication, an evolution within research about ICT in mathematics education should be visible when comparing them. We found common features in these two papers:

- they reflect on software applications and look for their benefits in the learning of algebra,
- they emphasise the potentialities of ICT for visualising, offering multiple representations and generalising,
- they take into account the modifications that technological representations bring to mathematical notions,
- they are based on in-depth experimentation of situations built from this analysis, but without explicit link with 'ordinary' mathematics teaching.

We found also that the papers differed in some aspects:

- Kieran et al. refer their analysis of the knowledge to theoretical elaborations on the teaching and learning of algebra not specific to the use of ICT. They use it to look at how technology changes the practice of algebra and at possible obstacles that it could bring.
- Tall is principally interested in the effects of the experimental teaching. He measures this effect by means of external comparison (results at pre- and post-tests by experimental and control groups), whilst Kieran et al. privilege the observation of students' behaviour and solving processes.

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- Tall's approach starts from the potentialities of technological environments at a relatively general level. He offers experiments as illustrations of these potentialities. When difficulties occurred in their actualisation, the paper discusses how to overcome these difficulties. In contrast, Kieran et al. look at the potentialities by focusing specifically on the software environment that they developed and experimented with. Potentialities and limits of the new technologies are considered through the options they took when designing the software. Thus potentialities can be more directly questioned.

These two papers illustrate common approaches and different options that can be found in other papers of the corpus. Do they reflect an evolution in research about ICT in mathematics education, as we expected in the selection? Let us compare for instance Tall's paper with a more recent paper by the same team: Graham and Thomas (1997). Although this paper is about graphing calculators whilst Tall's was about computer programming, the two papers have many features in common. The same tasks and potentialities are considered and evaluated. The papers share a common statistical approach to the superiority of teaching and learning with technology.

Graham and Thomas (1997) show however an evolution in the didactical analysis of knowledge. The notion of 'procept' is important in the paper. It takes the dual nature of algebraic and calculus concepts into consideration. In this approach, a concept has to be treated and conceived as an object in some situations and as a process in others. For instance, the graph of a function has global properties (general shape, regularities, etc.) of an object. It has also to be considered as a process associating a dependent and an independent variables. Processes are encapsulated into objects and a subject knows about a concept by encapsulating as well as by decapsulating it. The important point is that, although this notion has been elaborated in the context of the use of ICT, the ambition of its creators goes beyond this context, offering the notion as a means to think of conceptualisation as a general level. Thus, in some aspect, Graham and Thomas meet Kieran et al. by linking their analysis of the use of technology to general approaches to the teaching and learning.

We discern also a relative evolution in the 'problématique' and in the role played by the experiment. Graham and Thomas as well as Tall are looking for evidence of better learning by way of an external measure. Graham and Thomas prepared the sessions with the teachers, tested the students before and after, but made no direct classroom observations. In their paper, they do not discuss the design of the learning situation and do not question the potentialities of technology whereas Kieran et al. did. We found nevertheless in this short paper a report of teachers' observations in order to account for the better learning. Thus, although the 'problématique' didn't really change, we discerned a new interest in an internal view into the situations of use of technology.

At a more general level, looking above at the sub-corpus of papers on the use of CAS, we observed some emphasis on questions, discussions on tasks, and a variety of theoretical approaches. This is different from the state of this field

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before 1994 (see Mayes, 1997) where comparative studies were a majority and backed up by 'ad hoc' theorisation like for instance the 'White box, black box' approach (see Berry et al., 1994). Thus, we do have indications of a relative evolution.

Cognitive Dimension

This dimension is widely considered in the corpus, although, as we mentioned before, few convergences appear. We selected three papers, each of these representing a cluster different in their approach.

The first one (Laborde & Capponi, 1994) has a classical constructivist approach and considers the influence of computer representations on learning. The 'problématique' of the second paper (Yerushalmy, 1997) is about visualisation. The third article (Hoyles & Healy, 1997) is interested in the construction of meaning by the use of ICT, and the problem of the contextual character of this meaning.

The three papers share a special sensitivity to the role played by perception in cognitive processes. This is certainly a strong specificity of research on the use of ICT. The three papers certainly also differ from papers in this field, especially at the beginning of the period of our study, which are characterised by a 'naive approach' where the visual potential of technology is emphasised and offered as means for improving mathematics understanding and conceptualisation per se.

Like most texts in the corpus the three papers are situated beyond this naïve attitude and question the potentialities of technology for visualising with regard to the learning of mathematics. Laborde and Capponi show how an interaction with dynamic geometry software gives the students an understanding of visual or mechanical constraints that their construction has to conform to. This understanding is not directly a geometrical conceptualisation but it opens new ways towards this conceptualisation because the geometrical theory might be an explanation or a model of the visual properties of dynamic geometry. For Yerushalmy, the semantics of asymptotes is not just in the graphic perception of asymptotic behaviour of functions. It is in the conjunction of graphic visualisation and symbolic representations, and in the mutual interpretations that each representation can give of the other. To favour the construction of meaning, Hoyles and Healy stress the need for students to focus simultaneously on actions, visual relationships *and* symbolic representations. They design a microworld to provide a help for this.

Like these three papers most recent publications tend to present the relationships between perception and conceptualisation in dialectic ways. The cognitive power of visualisation tools and the underlying cognitive processes are a matter of investigation rather than just assumptions. For instance, accessing geometrical knowledge is no longer presented as resulting from the rejection of some perceptive apprehension of geometrical objects, but from the ability of relying efficiently both on spatial and geometrical competencies. More emphasis is put on the

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characteristics of problems and situations which can foster the dialectic interplay between competencies of a different nature and thus contribute to the development of geometrical expertise.

The three papers also bear witness to a strong convergence of theoretical approaches in the predominance of constructivist views stressing the construction of meaning by students linking action and mathematical reflection. In this general constructivist frame, references are made to various theoretical approaches.

Laborde and Capponi draw on Brousseau's theory of situations (Brousseau, 1997), stressing the role of the a-didactical milieu in authentic learning situations. The notion of a-didactical milieu makes operational the general idea of construction of knowledge by students. In Laborde and Capponi's paper it is a theoretical tool which helps in grasping the modifications of the learning situation introduced by technology. Explicit references to a theory of situations are not common in the corpus as we will see in the analysis of the situational dimension. Often papers refer to a change of learning situations that they do not analyse more precisely and they assume that this change will necessarily bring the students more directly to mathematical thinking. Papers like Laborde and Capponi's show that it is not so simple a matter.

For Yerushalmy, "technology can provide an alternative to rote learning and automatic memorisation by supporting a guided enquiry learning environment that allow the construction of definitions and algorithms by students". In her view, visualisation and action are central in the construction of pre-calculus concepts by students, as means to make tangible mathematical entities interweaving their geometric, graphic and symbolic properties. This centrality makes Yerushalmy's approach come close to theories specific to research trends about technology such as for instance, the 'process/object' approach, or the notions of cognitive tool and conceptual reorganisation.

The paper shows students acting on mathematical entities, constructing qualitative reasoning and formulations related to graphic behaviour of functions and interpreting symbolic expressions and techniques. As a contrast to 'naïve' approaches, Yerushalmy does not consider that visualisation will make students' construction of meaning necessarily easier. Complexities partially hidden in the symbolic aspects of the notions will become apparent and the gain will be richer meanings rather that easier learning.

Action and visualisation are also important in Hoyles and Healy's approach to technology. Their specificity is in the connections between meanings developed outside school or in specific technological environments and scientific notions. It is representative of a trend of research interested in the construction of knowledge as a 'web of connections' between meanings that they consider to be necessarily 'situated' or 'contextualised' (Noss & Hoyles, 1996).

Beyond this trend, many other papers show a sensitivity for that 'contextualisation of knowledge'. By introducing some distance with respect to standard teaching environments and norms, research in the use of ICT tends to act as a window on the situated nature of knowledge and on its dependence on the particular context in which it is built and used.

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The three papers show a diversity of cognitive approaches, representative of the wealth of conceptual elaboration in the field of the use of ICT. Convergence appears at a general level in a sensitivity to issues like visualisation, construction and contextualisation of knowledge as well as towards dialectical approaches to these.

Instrumental Dimension

Research papers considering indicators of the instrumental dimension are few. Our statistical procedure helped to select two papers among them, one in geometry (Pratt & Ainley, 1997) and one in algebra (Chacon & Soto-Johnson, 1998).

In these two papers, no instrumental analysis appears explicitly. The observations made in the papers of the students' uses of technology can nevertheless be interpreted in terms of the theory of instrumentation.

The first paper draws on theorisation derived from research on LOGO to conceive and interpret the use of the dynamic geometry application Cabri by young students. It starts from a study of the organisation of objects and actions with Cabri in relation to the underlying theoretical geometry. The question is whether children will be able to understand this organisation when they have no access to this geometry.

In the experiment, children, not influenced by geometrical considerations, developed utilisation schemes (Verillon & Rabardel, 1995, p. 86) compatible with the 'required activity' in Cabri. At first children assimilated the items of the 'creation' menu of Cabri and drawing tools in a graphic software, and easily derived utilisation schemes of these items from schemes associated with graphic software. Then, the development of schemes related to the 'construction' menu strongly depended on the goal assigned to the children's actions. In the first case, the children's goal was just drawing, and schemes related to the 'creation' menu were functional enough. 'Messing up' the drawing by moving components was a surprise to the children, but it did not invalidate their realisation, because it was compatible with the 'simple drawing' conception that they associated with the software. In another case, children had to design a 'drawing kit' and it was essential that properties of the objects remained when they were moved. Children had to build an understanding of these properties as invariants associated with items of the 'construction' menus. This accommodation was done through a synthesis of utilisation schemes related to drawing tools and knowledge built in the practice of LOGO.

For us, the two cases are different instances of a process of development of utilisation schemes and therefore two different 'instrumental geneses' (Verillon & Rabardel, ibid., p. 96). The teacher's options played an important role in these geneses. When (s)he gave a clear purpose to children' use of the software and when (s)he explicitly introduced children to the logic of the software, the genesis was richer than when (s)he just let the children use Cabri to draw freely.

The other paper is in a trend of research in North America aiming to show

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the benefits that technology brings about in the learning of algebra at collegiate level. In contrast with many papers in this trend, it shows evidence of difficulties. Students often did not perceive the support that technology brought to their learning. The explanation given by the authors is that they had no indicators for that. It refers to the institutional dimension (see below): students worked in a computer laboratory in addition to the ordinary classroom and not with the ordinary teacher, and they did not see a clear link between the tasks and techniques with the computer, and the usual mathematics. We can also conceive that the genesis was not what the researchers expected. Difficulties in the use of the computer influenced students' perception of technology, and instrumental schemes built in the computer laboratory were of little use in the ordinary classroom. Going further in the instrumental analysis of this paper would be interesting, but not feasible because the paper gives too little data on students' behaviour and representations.

These two papers illustrate the variety of publications in this cluster where 'instrumental' concerns appear. In some papers, it is relatively easy to find data to do an instrumental analysis and in others this concern is of only marginal importance.

Institutional Dimension

Our statistical procedure showed a small cluster where indicators of a reflection on the insertion of technology into school institutions appear: the development of techniques related to the use of ICT, and articulation of these with paper/pencil techniques, together with analyses of the task offered to students in relation to the ordinary curriculum. These indicators converge in a concern for the viability of ICT in the teaching and learning of mathematics. Two papers were selected that are reflecting on this question. One (Artigue, 1998) is a synthesis centred on teacher development in the use of ICT. Questions about this viability follow from this 'problématique'. The other paper (Graham & Thomas, 1997) starts from the poor integration of ICT and stresses new tools (the graphic calculators) that it assumes to have a better viability. We already considered this paper above.

The two papers start from pragmatic concerns. Artigue's aim is to reflect on what should be teacher professional development in the use of technology. Graham and Thomas want to show the benefits of graphing calculators to teach the beginning of algebra. In both papers, these pragmatic approaches induce the authors to consider the viability of computer applications in the teaching of mathematics.

In the first paper, the concern for viability derives from the author's view that optimistic assumptions about benefits of the introduction of technology are questionable. Her concern goes therefore beyond teachers' development to consider the obstacles standing in the way of the integration of technology. These obstacles are created by the lack of reflection on possible functions of ICT in the teaching of mathematics currently dominated by the use of traditional

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instruments (paper/pencil). In the author's view, awareness of these obstacles has to be a goal of teachers' development.

In the other paper, graphing calculators are assumed to be a means to remedy the difficult viability of computers in classrooms. We analyse this as a typical phenomenon in the development of technologies in school. When a new technology appears (in this case the computer and software with variables) pioneers make assumptions, often optimistic ones, about the benefits that learning can draw. Then, as years pass, the poor progression of the use in schools has to be recognised and unexpected difficulties become evident. A generalisation of the use of technology in schools appears not to be at hand. But, as the viability of technology is questioned, a new generation of technology appears and pioneers can again work on new assumptions. So, Graham and Thomas' paper mentions the difficult viability of computers in school only to show how a new technological generation (the graphic calculators) can be a solution.

Some years later, it is possible to question Graham and Thomas' paper regarding this issue of viability. Yes, as technological objects, the calculators are easier to use in classrooms than computers, but comfort of use is not the only factor of viability. Even with a calculator, tasks involving the treatment of variables by the calculator might appear too far from the ordinary learning of algebra, which involves much 'work by hand' that teachers might privilege in the reflection on algebraic expressions. The concern of teachers, reported at the end of the paper, for making more connections between the tasks with the calculator and the ordinary tasks and learning of algebra gives clear evidence that the viability of the tasks offered by the authors is not guaranteed by the use of calculators. The difficulties of students in Chacon et al. (see above) to perceive the support that technology brought to their learning also shows us evidence that it is not easy to insert knowledge built through the use of technology in the ordinary learning situation.

Like the preceding cluster, this 'institutional' cluster is composed of papers with very varied approaches to institutional issues. In Artigue's paper this concern is so central that it puts the initial subject (teacher development) in the background. Other papers start from this concern to motivate the experiment of a new technology and others just mention possible difficulties in their conclusion.

We see also that, to address questions related to the teacher, Artigue's paper has to consider them inside more general issues. This illustrates a situation typical of the years 1994–1998 when questions about the teacher brought about more general problems: how, for instance, to initiate a reflection on teachers' development when so little is known about what could be a real integration of technology? The deceptive outcomes of the statistical analysis that we got for this dimension corroborates this view (see above).

Situational Dimension

A small cluster deals with some aspects of the influence of ICT on learning situations. At the centre of this cluster, we selected a paper (Koedinger &

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Anderson, 1997) not directly belonging to Mathematics Educational research as representing 'AI-ED' (Artificial Intelligence in Education). This field of research works on models of learning situations with the prospect of their implementation into computer applications. As part of research and innovation on the use of TICE in mathematics, this field offers software environments with the ambition of promoting a student-computer interaction as a learning situation in itself.

We will also use in this section the paper by Laborde and Capponi (1994) that we considered above in the cognitive dimension. This paper relies on a theoretical approach to learning situations in order to reflect on the changes that technology brings about. These two papers will give us a hint about the variety of the approaches in this dimension, as well as the problems that a study of the learning situation raises when introducing ICT.

Koedinger and Anderson (1997) report on an experiment with an intelligent tutor in schools in some 'big cities'. Beyond the statistical evidence of an improvement of learning, this insertion of software tutors into the ordinary classrooms outside research laboratories is worth noticing. There is not much data in the paper on how the tutor was integrated but the reader can see that researchers had to think about this integration by adapting the curriculum and creating new learning situations. This is why the paper is at the centre of a cluster concerned with learning situations.

At a general level, the authors admit that students' interaction with the tutor is not sufficient in itself and must come together with situations of communication between pairs where writing must be privileged. At a more specific level, the analysis of student-tutor interactions shows how the 'economy' of the solving process is changed. The 'flagging' of errors allows the tutor to give the student a timely feedback and to evaluate his/her progression. Help is permanently available, tuned to the needs of the students and organised on several levels, from a brief hint to a full explanation of the problem. The 'knowledge tracing' uses what the tutor knows about the student from preceding resolutions to offer him/her a relevant succession of problems.

In these two aspects of the insertion of the situations of interaction into other situations and of the role of the tutor, the paper is concerned with the influence of technology on the learning situation. Other aspects remain in the shade. The first one is the role of the 'knowledge' that the tutor has in the domain of the problems offered to students and on the students themselves. This knowledge certainly helps the student but it also brings more fundamental changes, depending on how the student will integrate this knowledge into his/her strategies of resolution. It also changes the function of the teacher, an issue that the paper does not address.

As mentioned above, Laborde and Capponi draw on a theoretical approach to the situations, and an epistemological analysis of the knowledge, as tools to analyse the learning situations with technology. In their conclusion, they stress the characteristics of the new 'milieu' provided by dynamic geometry. Procedures based on pure drawing or empirical adaptation are easily disqualified but this does not make the students directly aware of the geometrical thinking requested

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by the task. What the students get is an understanding of visual or mechanical constraints that their construction has to conform to. This understanding can be seen as an outcome of the perception of moving Cabri objects in an a-didactical situation as a contrast with the 'drawing conception' resulting from paper/pencil practices.

As the authors note, this understanding of visual or mechanical constraints gives way to a new relationship between visualisation and geometrical thinking. In paper/pencil settings, geometry is built not to explain visual observation, but to make up for the limits of the perception. The relationship between visualisation in dynamic geometry software and geometrical theory is more towards explanation and modelling than break-up. It opens new ways of conceptualising geometrical certainly not as direct as in the initial hypothesis, but likely to result in a process in which students become gradually aware of the underlying geometrical properties.

Laborde and Capponi's paper is an isolated instance of the use of a theoretical approach to analyse changes in learning situations. Koedinger and Anderson's is more representative of the small cluster of publications that the statistical analysis showed. This cluster bears witness to an interest in learning situations and also to a lack of precise analysis of these situations. Papers in this cluster put forward some changes brought about by technology: a new economy of the resolution, changes to be made in the curriculum, and the need for thinking about the use of technology together with other learning situations. They rarely question the new strategies (not necessarily nor directly positive) that the students develop in these situations, nor the new role of the teacher.

SUMMARY AND INTERPRETATION

We saw the huge literature on the use of ICT for teaching and learning mathematics as data representing the efforts of a varied community of teachers, innovators and researchers. A quantitative analysis of this global corpus confirmed a gap between the literature and what we assumed to be the actual technological environment of an average student, at least in developed countries: the papers tended to consider computer software or symbolic calculators more likely than ordinary calculators or emergent technologies (CAI on the Internet or on CD-ROM).

Then a qualitative analysis showed the variety of publications. We found a wide range of approaches, from innovation, working on the most recent developments and providing potentially interesting contributions on the use of up-to-date technology, to didactical research that we saw as elaborations from these contributions.

Didactical research was found in two privileged directions. One direction was to look for evidence of better learning. In this case, like innovators, the researchers focused on assumptions about the benefits of recent technology. As a contrast with innovators, they referred these benefits to an analysis of knowledge, and provided an external comparison between a control and an experimental group.

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Another direction was to investigate questions about the modifications that technology induces into learning situations. The range of possible concerns was then wider and took the complexity of the educational situations more into consideration. The argumentation was based on the internal observation of teaching and learning situations and was backed up by an analysis of knowledge issued from didactical literature.[5] The statistical procedure that was carried out on the research papers clearly separated them into these two directions.

Innovative papers offer a wealth of ideas and propositions that are stimulating, but diffusion is problematic because they give little consideration to possible difficulties. Didactical research has to deal with more established uses of technology in order to gain insights that are better supported by experimentation and reflection. We have then to think of these two trends as complementary rather than in opposition.

Beyond a relative convergence, the two directions that we found in research will, in our opinion, persist as two different styles of research. The first direction aims at having a short term influence on curricula and educational policies. External comparisons and specific theoretical elaborations are seen as means to provide sufficient evidence and support to make decisions. Although a part of research, this direction is close to innovation because of its pioneering spirit. We noted at the beginning of this chapter that such evidence has to be seen in relation to the limited numbers of factors of integration that researchers can consider. Outcomes of comparative studies can then be considered not really as findings, but rather as potentialities of technology. The second direction is more classical, in keeping with the general style of educational research. Some papers in this direction are interested in technology as means for a general understanding of learning situation more than in the possibility of a generalised use. Papers interested in this use generally look for long term effects.

Our view of the literature on the use of ICT is then of a 'three stroke cycle'. Innovation produces situations of use. Comparative research papers investigate these situations in order to get evidence about their benefits. These benefits – or more accurately potentialities – of technology provide material for research studies focusing on the understanding of learning situations or on long term effects.

The 1994–1998 literature appeared to restrict its analysis to potentialities of ICT itself (easier and more varied representations, new aspects of mathematical knowledge, etc.) rather than questions raised by its insertion into the 'ordinary' mathematics teaching. Despite this restriction, it provided interesting material when phenomena observed in this period could be interpreted in new dimensions, whether instrumental or institutional. As compared with today's literature (see Mariotti, to appear), we found the instrumental dimension to be in an embryonic state. Notions like utilisation schemes and instrumental genesis are not explicitly mentioned. They emerge as a means to understand the influence of meanings that students built using technology.

The main institutional concern that we found was the difficult viability of technology in schools. This concern was however shown through very varied

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approaches. Papers with a pioneer spirit started from today's difficulties to motivate the use of tomorrow's technology, while others looked for reasons in more permanent characteristics of technology and of the educational institutions. The relationship between ordinary paper/pencil work and the use of technological tools was an emerging issue. To address this issue, no real theoretical elaboration was found in our corpus when today the 'techniques' are seen as an important level, intermediate between tasks and conceptualisation, and this level is taken as central in the relationship between ordinary and instrumented work (Kieran, 2001).

In spite of the constructivist reference of many papers, situations of integration of technology were rarely completely analysed. We found one isolated instance of the use of a "theory of situations" (Brousseau, 1997). In recent years, the interest in this theory has grown in the international literature and it could help when looking in depth into changes in the learning situations and when showing precisely what is at stake in these new situations (Sutherland & Balacheff, 1999).

Apart from the constructivist references, we found a rich variety of cognitive frameworks. Elaborated primarily to account for 'laboratory' experiments, they are consistent with the focus of literature towards epistemological concerns. Elements of evolution appear in the convergence towards dialectical approaches to issues like visualisation and contextualisation. These approaches contribute to the development of new dimensions by helping to better consider the institutional contextualisation of knowledge as well as the schemes attached to the use of a technological tool in their instrumental dimension.

In the years 1994–1998 questions about the teacher necessarily brought about more general problems with few solutions. There was a tendency to focus on teachers' development and an implicit assumption that the transfer of innovative situations of use, possibly supported by outcomes of research, would provide the teacher with sufficient material for an easy integration. Aware of the complexity of teaching and learning situations with ICT, researchers are now more cautious. Interesting research studies start from the observation of teachers struggling to integrate ICT into the real teaching (Monaghan, 2001). The study of contrasting teacher decisions helps to consider constraints inducing teachers' privileged views on the use of ICT (Kendal & Stacey, 1999).

The role of the teacher in students' instrumentation appears as a key question. How teachers take into account the use of instruments by students has rarely been addressed. With paper and pencil this role of the teacher is generally not considered to be relevant in the analysis of students' learning. Evidence exists of very different students' instrumental geneses resulting from varied teachers' options, and of a strong influence of these on students' conceptualisations. This evidence will make researchers consider the teacher's action on instrumentation as an important dimension in his(her) activity.

To conclude this overview we consider again the motion of innovation, pioneer and classical research. This is a fast motion following the rapid evolution of technology. In the years 1994–1998, it produced analyses centred on epistemological issues and the learner. Its influence on classroom practices was however

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deceptive. Beside this short cycle, we discerned a long term motion towards awareness of a more complex integration and the subsequent necessity of new dimensions of analysis. It is confirmed by what we know of the institutional and instrumental dimensions in today's research studies and of the emerging reflections on the teacher. We hope that the multidimensional framework we have presented in this paper will facilitate this motion by providing innovators and researchers with a set of references to deal with the complexity of teaching and learning with ICT.

ENDNOTES

1. An annex summarises ten of the seventy-nine reviews. We are grateful to a number of colleagues who participated in this research particularly by establishing these reviews. We thank especially H. Chaachoua, E. Delozanne, D. Guin and B. Grugeon whose reviews have been very useful for this paper.
2. 'Problématique' comes from the German 'Problematik' which means "the art of posing problems".
3. Including papers about symbolic calculators (6%) and about CAS software (21%): see above.
4. This journal was first published from 1994 under the name 'International Journal DERIVE'. In 1997 it changed to 'International Journal for Computer Algebra in Mathematics Education'. The editor is John Berry and the publisher is Research Information Limited (UK).
5. 'Internal' versus 'external' refer to the French tradition. This is certainly close to the distinction between 'interpretive' and 'quasi-experimental' studies found in US surveys, for instance, Wilson et al. (2001): "by interpretive' studies, we mean those that try to understand educational experiences from the perspectives of those involved (...)".

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ANNEX: short reviews of papers of the 1994–1998 corpus analysed in the chapter

ARTIGUE, M., 1998

Approach

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The starting point of the paper is that, in spite of the efforts of school institutions and of the militancy of pioneers, computer applications exist only marginally in the teaching and learning of mathematics and their integration progresses very slowly. The author's view is that the usual training does not make teachers aware of obstacles standing in the way of the integration.

Study and results

Research studies raised issues that teacher training does not take enough into consideration. The author gives an account of these:

- Potentialities of ICT don't ensure legitimacy in the teaching and learning. Even when teachers are sensitive to the impact of computers in society, they are not easily convinced of the usefulness of technology in their own teaching. Then, promoters of ICT tend to overemphasise the potentialities and minimise the actual difficulties that teachers will meet. In this kind of vicious circle, integration cannot reach maturity.
- Computer transposition has unexpected effects on mathematical knowledge. Promoters underestimate these effects. They tend to hide the fact that they cannot be avoided, together with the resulting difficulties. In contrast, when phenomena linked to this transposition are recognised and understood, they are a valuable basis for mathematical reflection.
- The traditional opposition between the technical and conceptual dimensions of mathematical activity is unproductive. ICT are often offered as tools get rid of the burden of technical aspects in the resolution of problems and to favour a conceptual view of solutions. Research studies gave evidence that ICT can also favour blind action and inconsistent atomised solving process. To avoid this, a balance between the technical and conceptual dimensions is not to be missed, but teacher training does not give the teachers means to understand what this balance might be.
- The instrumental dimension is important for the mathematical conceptualisation. The teaching of mathematics usually works in an environments without rich technology. Integrating ICT changes radically this environment. Teaching should consider students' transformation of technological tools into mathematical instruments and the associated processes of instrumentation and instrumentalisation.

Obstacles on the way to the integration come from insufficient account being taken of these issues. Teacher training should then focus especially on didactical reflection that could help teachers identify key variables and to analyse their professional activity and the changes they will meet when using technologies.

CHACON, P.R. & SOTO-JOHNSON, H., 1998

Approach

The paper refers to research studies on the effect of computer environments on students' work and conceptions. These studies tend to give evidence of positive effects on the learning of algebra at the collegial level.

The paper compares the effects of two courses in algebra at collegial level: one is

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'traditional' and the other includes a module with computer use. This module was one hour per week, students working on exercises and problem solving. The tasks involved the association of graphic representations and of algebraic expressions.

Study

The courses were offered to eight classes of twenty students. Each student had a graphing calculator. Six classes were the control group following the traditional course. In three of the six, the teachers asked the students to use the calculators. The two other classes were the experimental group using the computer module. Teachers in these classes also made the students use the calculators in other parts of the course. The methodology used pre-, post-test and recorded interviews. Researchers intended to evaluate the evolution of students' view on technology and on mathematics. They also tried to make students report on how they acquired knowledge through this course. The interpretation of data was controlled by a statistical procedure and supported by excerpts of classroom discussion and interviews.

Results

Two differences appeared between the groups. The experimental group got more proficient in the graphic and symbolic analysis. This group also developed a more critical attitude towards technology. Students perceived benefits of the use of computers, but also reported difficulties they had because of the module. The authors' explanation of students' possibly deceptive views of the use of technology were:

- the exercises that students did in the module with computers were different as compared to the ordinary teaching, and students were confused,
- computers were available just during the module and thus students felt frustrated with not being able to use computers at their own will.

The authors' reflection was that it is not easy to evaluate what students actually learnt using the computer and thus it would be necessary to elaborate new indicators of students' understanding of fundamental concepts that would be relevant for the use of technology.

GRAHAM, A. & THOMAS, M., 1997

Approach

This paper belongs to a series of studies by a team led by Tall and Thomas addressing the issue of computer applications in learning algebra. It focuses on the notion of variable in the early learning of algebra.

The paper takes as a background varied theoretical elaborations about the learning of algebra: for example, the notion of procept is discussed with regard to classical approaches to the learning of algebra in mathematics education.

From this, and from previous experiences, the authors assume that, by using computer environments involving variables, students can handle numerical examples to make and test algebraic conjectures, and then get a mathematical experience and a higher level of abstraction.

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With regard to these potentialities, the authors are concerned by the slow integration of computers into mathematics classrooms. A reason for that is the lack of appropriate computer hardware as well as relevant and reliable software. The authors assume that things will change because of the availability of hand held calculators. Calculators make intrinsic use of variables in calculations and their easy manipulation allows multiple interaction and experimentation. They suggest that teachers should take advantage of the availability of calculators to actualise the potentialities that they found in previous research studies about the use of computers.

Study

Experimental classes used the TI-80 calculators during three weeks. The students were 12 to 14 years old of mixed or high abilities. Most students had no previous use of calculators. In the early exercises students had to store values into variables and predict the values of algebraic expressions involving the variables. Then in another task, students were given screen views involving variables' names on paper and were asked to reproduce these on their calculators.

The experimental students were compared to control classes taught using usual teaching methods. The data was made from students' writings in pre- and post-tests. Previous research studies were used to design the tests. A statistical treatment was designed to investigate how experimental students understand the concept of variable as unknown and generalised number, as compared with control students. The experimenters conceived the tasks and the associated curriculum, collected comments from students and teachers about the experiment, but didn't directly observe the classroom sessions.

Results

The statistical treatment showed the expected improvement. Students commented that they enjoyed the novelty of the exercises. The comments of the teachers were most encouraging. Teachers also stressed the apparent lack of relationship between the calculator work and the manipulative work.

HOYLES, C. & HEALY, L., 1997

Approach

The aim of the paper is to analyse the processes through which students come to negotiate mathematical meanings for reflective symmetry. The processes involve students' interaction with a computer based microworld 'Turtle Mirrors'. Following Harel and Papert (1991), a microworld is taken in the original sense of a world simultaneously rich and simple enough to study learning behaviour (not necessarily human).

The rich set of meanings around reflective symmetry developed outside school is an opportunity to make connections with scientific notions introduced in schools. New understanding can then be created by reference to context and action as well as by systematisation and formalisation, and meaning is taken as deriving from a 'web of connections' (Noss & Hoyles, 1996) constructed as a result of the dialectical interaction of action and thought. A microworld might favour this construction by helping students to focus simultaneously on actions, visual relationships and symbolic representations.

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Study

A microworld was designed from a study of students' behaviour and misconceptions when faced with a task of completing the image of a pattern symmetrically with regard to a 'mirror line' in various settings. The general design of the microworld involved two turtles (a blue and a red) animated by means of LOGO instructions. So a student was able to act on the orientation and motion of the turtles as well as on the mirror line using symbolic angles and lengths and visualising the turtles' respective movement.

A detailed case study of one student's 'thinking-in-change' is offered. Her behaviour in the paper/pencil tasks before and after using the microworld, and her interaction with her partner and the computer when working in pairs with the microworld are reported and analysed. It is completed by other examples of students' work.

Results

The analysis of the data showed that the microworld offered students a way of talking about and operationalising the angle and distance properties of reflection. Students were able to think of the reflection from turtles turning from a heading along the mirror line in opposite directions and going the same distance. This meaning is operational even when students have to achieve a paper/pencil task. It is an empowerment because it gives the student a mental tool which can be used in new problems situations. The meaning is nevertheless 'part-and-parcel' of the activity in the microworld and therefore possibly limited, because it makes students' thinking closely dependent on a technological tool.

An example is given where a student was able to build a solution to a paper and pencil task by recognising a property of the turtle motion that she was not aware of when working in the microworld. So the meaning built in the microworld appears to be flexible and thus not far from a mathematical conceptualisation. This suggests that learning in the context of a computational microworld is not so questionable about meanings too closely dependent on the technology. The abstraction resulting from this learning might be a valid mathematical conceptualisation even when it is 'situated' in the microworld. The real question would be that "there is little connection and no obvious path" from this abstraction to the official geometry.

KOEDINGER, K. & ANDERSON, J., 1997

Approach

The paper starts from a 'cognitive' model designed to be implemented in a computer application. This model is a set of production rules, generating the steps of a resolution (right or wrong) that students might find. For a given problem, this model helps to foresee students' resolutions and to manage the progression from one problem to another. It also includes an 'intelligent tutor' as a guide for the student. The design of the model separates the authors' expertise in artificial intelligence and cognitive psychology, and the teachers' knowledge in curriculum development and in classroom practice. The teachers in the school experimenting with the project are thus considered as 'clients' in this design.

Study

The paper reports on an experiment of the computer application together with a curriculum for learning algebra. This curriculum is centred on modelling using functions and

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the use of varied representations (tables, graphics, symbols). Students worked in team on mini-projects. The work on the computer went with classroom sessions aiming at the transfer of proficiencies acquired with the computer to paper/pencil techniques. These sessions focused on writing by way of doing reports and communicating findings to other students.

The experiment is a comparison between experimental groups using the computer application (470 students, low abilities) and control groups with a 'traditional teaching' (155 students, mixed abilities). The comparison uses standard tests for assessing algebraic learning and other tests more consistent with the new curriculum.

Results

A statistical treatment showed that experimental students scored 15% better than control students in standard tests and 100% better in the other tests. The authors also report that teachers and students were enthusiastic about the experiment.

KIERAN, C., BOILEAU, A. & GARANCON, M., 1996

Approach

This paper draws on research studies about teaching and learning of algebra and about the interest of graphic representation to support mathematical formalisation.

The computerized environment CARAPACE developed in this research project is based on an introduction to algebra based on functions. It offers a problem solving environment which includes three representation modes: 'algorithmic', table of values, and graphical representation. The 'algorithmic' representation is a kind of natural language representation rather than a traditional symbolism. Letters are considered more as variables than as unknowns.

Study

Successive versions of the CARAPACE software have been used with groups of 6th to 9th graders between 1987 and 1994, without the presence of a teacher. The students' writings were collected and their behaviour was observed. The paper analyses cases of students learning the first algebraic notions using CARAPACE. It investigates how solving procedures as well as the status of letters and manipulation rules in this environment change during the interaction. The role of the different modes of representation is also studied.

Results

The entry into algebraic thinking from the notion of variable (i.e. letter with a set of numerical values) makes students approach the notion of an unknown (i.e. a letter with one or several values verifying a given condition) in a broader context: they progress without any difficulty from variables to unknowns.

Using CARAPACE, the students do not become spontaneously proficient in algebraic manipulations or in the expression of families of functions in varied forms. They actually favour a numerical approach, showing difficulties when considering the parameters of

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linear or quadratic expressions at an abstract level. But this numerical approach does not become an obstacle.

The CARAPACE functional approach mobilises thinking processes which are parts of traditional algebra (global representation of the problem, generalising from examples). But it does not take into account the written manipulations that make algebra more than a tool for problem solving. CARAPACE cannot contribute to making algebra appear as a theory giving sense to the notions of unknown, variable, function, equation, algebraic transformation, etc., nor as a tool for mathematical modelling of situations.

The new environments (calculators, spreadsheets, CAS, etc.) use different kinds of representation (registers) which are not easily accessible for the pupils. There is therefore room for pedagogical software devoted to specific learning.

LABORDE, C. & CAPPONI, B., 1994

Approach

This paper is centred on the learning of the notion of geometrical figures, defined as a set of relations between a geometrical underlying object and the attached drawings. According to the authors, this relationship has to be built by the students through an organised learning, and this learning is more likely to occur when students are faced with situations in which a geometrical analysis of the drawing is an efficient tool of solution. Following Brousseau's theoretical approach, the authors stress the need for an 'a-didactical component' to provide authentic learning situations.

In these situations, students should go beyond a perceptive level and interpret drawings as representing geometrical figures. The problem is to be designed so that strategies for solving based on geometrical knowledge will be more efficient than empirical perceptive strategies and to be found by the students as real means to solve rather than as a way to comply with teacher's expectations.

Focusing on problems requiring a geometrical construction, the authors' hypothesis is that the dynamic geometry software Cabri-géomètre can be a component of such an 'a-didactical milieu'.

Study

The paper considers tasks in geometry first in ordinary paper/pencil settings and then in a Cabri-géomètre environment. Students' tasks in the paper/pencil settings involve the selection of geometrical information about drawings as well as the use of traditional drawing instruments, possibly supported by a geometrical reflection. It is actually difficult in these settings to anticipate the students' reflection, because there are ambiguities about what information may be selected and what method of construction is to be accepted. A geometrical reflection can thus be expected only when the teacher asks for a discursive characterisation of the geometrical object attached to the construction which doesn't keep the situation a-didactical.

Constraints on instruments are often proposed as means for a better a-didacticity of situations based on problems of construction. When not allowed to use familiar instruments, like for instance a ruler for measuring, students should reflect on the geometrical properties underlying the use of the allowed instruments. But again, there is no strong guarantee of a-didacticity because students might use instruments in unexpected ways

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and when the teacher gives directions on how to use the instruments (s)he also drives the students towards the solution.

Dynamic geometry software like Cabri-géomètre are designed to favour a geometrical reflection though a-didactical situations because such pieces of software require the students to communicate a construction rather than just do a drawing. The Cabri construction itself is not just a drawing, because its geometrical properties do not vary when basic elements are moved. Thus, using Cabri just as a drawing tool should be invalidated by the software, helping students to switch towards the use of geometrical software functionalities.

The authors' hypothesis is that, in a task of construction using Cabri-geomètre, the students will interweave a perceptive activity (drawing, moving, etc.), strategies of combinations of Cabri functionalities and the use of geometrical knowledge. The restriction of menu entries should give the teacher means to control the students' strategies and knowledge, in more authentically a-didactical situations, as compared with the above paper/pencil settings. A situation for 8th grade students is offered as an example. An a priori analysis of the resolution is done and followed by a report and analysis of the resolution of students in two classes.

Results

The a priori analysis helps to evaluate the possible students' strategies. Strategies based on random combinations of menu items cannot be directly successful, but a random trial of menu entries might give way to a geometrical reflection and a solution. Students' resolutions started by perceptive drawing, using the items of the 'creation' menu. Moving objects helped to reject these strategies. After that, the students tried the items of the 'construction' menu, first at random. Then, moving objects helped them to recognise resistant properties and to think of a consistent resolution.

What the students get is an understanding of visual or mechanical constraints that their construction had to conform to. This understanding can be seen as an outcome of the perception of moving Cabri objects in an a-didactical situation as a difference with 'drawing conception' resulting in paper/pencil practices.

PRATT, D. & AINLEY, J., 1997

Approach

The authors start by considering the absence of geometry teaching in UK and analyse the reasons for this situation: deductive geometry tended to ossify, the 'new maths' approach via transformations was unsuccessful. In this context, and in absence of a formal teaching of geometrical concepts, the use of LOGO by pupils led to many examples of creative and imaginative work and gave rise to the development of interesting theories about learning (Harel & Papert; Noss & Hoyles).

The core issue of the paper is the design of new situations: in these situations students using a geometrical package instead of LOGO should develop the same creative work; theories based on LOGO experimentation should help to consider these new situations. Dynamic geometry software is analyzed according to the distinction between drawing and figure and possible interpretations of software tools (basic objects, functions, constructions). The paper aims to analyse how young children (8 to 12 years old) without any

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formal geometrical knowledge use the software program. Therefore two contrasting cases are considered.

Study

In these two cases, each pair of pupils had at their disposal (at home and at school) a laptop for two terms. In particular they used a graphic module of a software package. The first case study is about children (8–9 years old) spontaneously using Cabri as a drawing software tool. The researchers questioned the children about their use of Cabri and analysed what the children did. The second case study concerns older children (11–12 years old). This was a more formal introduction to Cabri in a teaching situation. Pupils were asked to construct a drawing kit meant for other pupils.

Results

In the first case study, almost all children spontaneously used only the 'Creation' menu. Only one child succeeded in using an item of the Construction menu, but without giving a sense to this action. Children used Cabri as a drawing tool, exactly as they used the other one they knew. Then the researchers moved the free points of the drawing to show that the geometrical relationships were not preserved in the drag mode (they 'messed up' the drawing). But it did not make the pupils question the way they used Cabri.

In the second case study, the introduction of the software program by the teacher (in a situation of an equilateral triangle construction) seemed to have an effect. The purpose of the activity (building a drawing kit) also led the pupils to give a meaning to the preservation of geometrical relationships. Many links with LOGO were observed in the pupils' verbal exchanges, namely between LOGO procedures and CABRI macros, between the 'stamper' action (specific LOGO vocabulary) and the relationships between objects created from Cabri constructions. The dependency between objects in Cabri was grasped by a group of children through their own experience by erasing objects that they wrongly considered as not useful. Cabri in such a case erases the object and all objects depending on it. Thus, in a short time, through feedback provided by Cabri (dependent objects surprisingly disappearing) pupils realised what a Cabri construction is.

TALL, D., 1993

Approach

According to the author, technology opens a new way to learning because it frees students from certain tasks and allows them to focus on goals. Computerised environments also allow various entries for learning a concept. Visualisation (graphic representations, images) favours conceptualisation processes. Computerised microworlds help the learners to generalise by manipulating generic objects which to some extent represent the concepts. They are called by the authors 'generic organisers'.

Study

Several experiments are referenced:

(1) A 1988 experiment of the introduction of algebraic symbolism by way of programming.

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Students used the BASIC computer language as an 'algebraic math machine' for several tasks like instantiating variables to compare algebraic expressions and acquire a better understanding of the equivalence, trying to compute 'faster than the computer', using the computer for double check.

(2) A 1990 study of the introduction of trigonometry using the computer.
(3) A 1990 study of the use of the Graphic Calculus software to help students learn about differentiable functions.
(4) A 1989 experimentation of the use of Solution Sketcher to make students solve differential equations by graphic representation of numerical solutions.

Results

The author observed a definite improvement of the experimental students' achievement. They overcame cognitive obstacles and acquired a better understanding of algebraic symbolism, including more varied aspects, especially in considering the algebraic equivalence of expressions and in solving inequalities and linear equations. They reached a proficiency in manipulation of expressions (for example computation of derivatives) identical to students working in a traditional environment. The author's interpretation is that once a conceptual basis has been built, less time is needed to acquire manipulative skills.

The author wonders whether the computerised environments that he considered in the paper, favour a generic or a formal thinking. He stresses that 'generic organisers' work at a generic level more than at the level of formal definitions. There are therefore limitations in learning in these environments. Students nevertheless acquire a conceptual foundation better than with a classical method. From this foundation, they will have to do a cognitive reconstruction in order to reach the formal level.

YERUSHAMY, M., 1997

Approach

The central issue in this paper is the link between a graphic approach of pre-calculus concepts and the symbolic manipulation of expressions. The study of asymptotes is an interesting domain to investigate this issue. Graphic representations help to visualise and interpret the asymptotic behaviour of functions. In contrast, symbolic techniques of transformation of expressions are often seen as meaningless recipes to get equations of asymptotes.

In the paper, graphic observation is done with a specific software tool (the 'binary calculator') linking symbolic combination of functions (sums, products, quotients) to their graphic representation. The question is then how students can interpret the symbolic techniques of transformation using knowledge about asymptotical behaviour that they get through graphic observation.

The paper refers to didactical research about the conceptualisation of limits and about misconceptions. It draws also on more specific theorisation: technology makes mental objects tangibles by offering them as virtual objects.

Study

The study concerns thirty students during eight sessions. In each session, students worked at first in small groups on computers and then there was a classroom discussion.

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Students had first to search for asymptotic behaviour of functions by graphic observation. Then they had to build functions whose behaviour is given. The final task was to express definitions of particular behaviour. Symbolic techniques (calculation of discontinuities and limits, fractional expansion) were given to students as tools to get more precise equations of asymptotes. The classroom discussion was about the interpretation by students of the results of these techniques.

Results

The students' writings and excerpts of classroom discussion are the starting point of a discussion about:

- Students' elaboration of qualitative reasoning and use of symbolic techniques. Teaching supported by the software tool as described in the paper can favour an approach where reasoning and techniques are interwoven in a rich reflection, very different from teaching strategies based on the execution and memorisation of procedures. Visualisation is thus not considered in the paper as making learning easier but as a basis for a mathematically richer activity.
- The understanding of symbolic techniques both in their syntactic and their semantic aspects. The semantic is made of action ('touch') and of perception ('see'). Graphic visualisation with the software tool helps to develop an understanding of this semantic, because students can manipulate the symbolic expression of functions and see the consequences on the graphical representations. It helps for instance to give symbolic manipulations a graphical interpretation in terms of asymptotic behaviour.
- The question of symbolic algebraic software. With this software, students could lose some control of symbolic techniques. On the other hand, the case of asymptotes shows that understanding is supported by a reflection on the outcomes rather than on the execution of techniques. Thus, the use of symbolic algebraic software could support efficient learning.

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