Complex calculators in the classroom: theoretical and practical reflections on teaching pre-calculus

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Abstract

University and older school students following scientific courses now use complex calculators with graphical, numerical *and* symbolic capabilities. In this context, the design of lessons for 11th grade pre-calculus students was a stimulating challenge.

In the design of lessons, emphasising the role of mediation of calculators and the development of schemes of use in an 'instrumental genesis' was productive. Techniques, often discarded in teaching with technology, were viewed as a means to connect task to theories. Teaching techniques of use of a complex calculator in relation with 'traditional' techniques was considered to help students to develop instrumental *and* paper/pencil schemes, rich in mathematical meanings and to give sense to symbolic calculations as well as graphical and numerical approaches.

The paper looks at tasks and techniques to help students to develop an appropriate instrumental genesis for algebra and functions, and to prepare for calculus. It then focuses on the potential of the calculator for connecting enactive representations and theoretical calculus. Finally, it looks at strategies to help students to experiment with symbolic concepts in calculus.

Introduction

Traditionally, computers and calculators are distinct technological tools in the teaching and learning of mathematics. Early computer use in mathematics teaching was through programming, but more recent use tends to favour use of generic packages including software dedicated to algebra or geometry. In the teaching and learning of algebra and calculus in the last 10 years there have been many experiments using Computer algebra systems, like MAPLE and DERIVE, see (Mayes, 1997). Over these years, the use of increasingly sophisticated hand held calculators has impinged on everyday life as well as on classroom activities. When sophisticated numerical and graphical capabilities were added, it became clear to students and sometime to teachers that calculators could play a role in solving problems involving functions (see Tall, 1996, Trouche, Guin, 1996).

New hand held calculators offer, to some extent, a synthesis of computer software and calculators[1]. Like computers they have powerful applications: computer algebra systems, geometric software and spreadsheet. From calculators they inherit ergonomic characteristics (small, disposable) and numerical and graphical utilities important to the study of functions.

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[1] This paper is based on a project where every student had a relatively expensive Texas Instrument TI–92. Manufacturers now offer complex calculators pricing like ordinary graphing calculators. For instance, the new TI-89 has the capabilities of the TI-92, except geometry, that we did not use in the project. Looking at the interface, the TI-92 is like a small computer (high resolution screen, alphabetical keyboard) and the TI-89 is like a graphing calculator. This difference is not very consequential for the discussion in this paper. Casio offers also a graphing calculator with symbolic capabilities (the GRAPH 80).

This paper presents an analysis of an attempt to integrate these powerful calculators in the teaching of pre-calculus in France. This integration has been carried out in four classes of the ordinary French scientific upper secondary level (11^{th} grade)[2]. In this paper, I do not seek to prove that teaching and learning with calculators is definitively better than with traditional paper and pencil. I merely assume that these calculators are legitimate means of doing mathematics[3]. From this assumption, this paper provides reflection, based on theory and practice, on the changes that these calculators may bring to the teaching and learning of mathematics, and a search for efficient means to use them in order that students learn meaningful mathematics.

This TI-92 experiment is a continuation of an earlier French experience looking at the integration of DERIVE into the study of algebra and calculus. Working in close co-operation with a group of teachers supported by the National Ministry of Education (DISTEN group, see Hirliman, 1996) to study the effects of this integration, we carried out a number of classroom observations from grade 9 to grade 12 (Artigue, 1995, 1997). We also questioned twenty five teachers and nearly five hundred of their pupils.

From this research we compiled a number of interesting insights on how technology may support the learning of mathematics, which will be referred to later in this paper. However, an important limitation of the DERIVE study was that students generally lacked the familiarity with this technology necessary to really use it to support their mathematical activities and learning. On many occasions, we saw students using their own numerical calculator to try to solve a problem numerically, when we expected them to solve it symbolically with the help of the computer algebra system.

So, when ‘computer-like’ calculators became available we saw the potential for easier student access to computer algebra technology which might affect their everyday mathematical practices, and that we would be able to observe more substantial changes. Therefore an offer by the National Ministry of Education to support a teaching experiment for pre-calculus 11^{th} grade classes where every student had a TI-92, was stimulating and welcome. However, from the DERIVE experiment, we knew that the integration of symbolic facilities into the work of the student was not an easy project. For that reason we had to develop a theoretical approach to this integration and design lessons/activities which could be applied in a wide range of settings: to students at various levels of attainment, with varied attitudes to calculators and mathematics; to teachers with distinctive epistemological views, teaching strategies and attitudes to calculator use.

It appeared that we had to reflect on two linked set of issues.

- First, how can we conceptualise changes in the mathematical activity in a classroom when every student has a powerful ‘computer-like’ calculator? To what extent can computer approaches in the teaching of mathematics be used? How does our experience of using computer algebra help us? What aspects of the work will be affected by the personal character of the calculator?

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[2] Michele Artigue was the leader of the team. Badre Defouad and the author participated with the teachers, Michele Duperrier and Guy Juge, to the definition of the sessions and did classroom observations and interviews. A report on the project can be obtained from DIDIREM Université Paris VII 75251 Paris Cedex 05, France. The project was founded by the French ministry of Education (DISTEN B 2). In the paper, ‘we’ and ‘our’ will refer to the team. ‘I’ and ‘me’ will be used to express my own ideas and work.

[3] Legitimacy of a technological tool is a complex question, which do not limit to improved efficiency. Use by professional mathematicians, acceptance by parents, allowance at exams are other important factors of legitimacy.

- Second, what conceptualisation of calculator use, with its many multilevel capabilities, arises in the teaching of a specific subject? What help do the numerical and graphical utilities bring? Regarding support for algebraic calculations, do the calculators help to build symbolic definitions of concepts? How can we think the introduction of the symbolic capability related with a concept (the key pressed to get a derivative or a limit...)? Does it help to students to conceptualise, if so in what way? Are these capabilities a danger ? Do students need to be at a certain skill or conceptual level before using tools like this?

These issues, concerning both technology and mathematics, are of general interest to those involved in mathematical education. The goal of this paper is to reflect on these issues and explore outcomes from real teaching situations.

The evolution of approaches to the use of computer technology in the learning of mathematics

Constructivist approaches

When computers became available, many hopes were placed on the autonomous cognitive activity that a learner could develop when faced with specific tasks (Artigue, 1996). The general frame was a Piagetian approach: acting in adequately problematic settings, the learner meets insufficiency or inconsistency of his/her knowledge. Introducing computer environment could help to create settings of this kind. The emphasis was put on the role of purposeful action in the conceptualisation of knowledge in opposition with the passive reception of meaningless mathematical contents. Computer tasks appeared well suited to these conceptions[4].

Another conception of the construction of mathematical concepts was easily adapted to computers: many concepts, especially in algebra and calculus, appear with two linked aspects, as a procedure and as an object. Gray and Tall (1993) introduced the name ‘procept’ to describe this duality in many areas of mathematics including calculus concepts. Computer activity, especially programming, can give a sense of this duality. A function, for instance, can be defined by means of a programmed procedure, then it will be considered and manipulated through the name of the procedure.

Repo (1994) reports on an example of this approach in the learning of calculus with the use of DERIVE. She blames the “quite algorithm oriented” learning of mathematics prevalent in Finnish schools, and offers six “critical activities” to activate prior knowledge of students, to internalise the concept of derivative, co-ordinate the representations of this concept, generalise it and understand its reversibility. I will briefly study Repo’s research because it had a significant influence on the view of Computer Algebra Systems as “cognitive and didactic tool to engage in reflective abstraction” (Mayes, 1997, p. 185). As for me, I see limits in this approach and considering these will help to adjust my reflections.

Repo’s research design is that of a comparative study: a control group received “standard mathematics teaching”, and the experimental group 50 lessons in computer room based on the above critical activities. In an immediate post-test, the experimental group performed significantly better on conceptual items, and in a delayed post-test it showed better retention of algorithmic skills.

My first criticism is that no evidence is given of the influence of the computer on these improved performances. The control group had a mainly algorithmic introduction to calculus,

[4] Papert (1980) is an influential example of a Piagetian approach in a computer environment.

and a common consequence is that they had a low understanding of calculus and a poor long-term retention of algorithms, so the better achievement in the experimental group may refer to the poor performances of the control group, rather than to the computer activities. Repo’s approach stresses an opposition between conceptual understanding and algorithmic skills, and the activities focus on understanding. Therefore the way students acquired these long-lasting skills is unclear. In France, approaches based on this opposition and on strong assumptions on the role of DERIVE to enhance the conceptual learning have been tried. I argue (Lagrange, 1996) that there is a gap between these assumptions and what actually happens in the classroom. Using symbolic computation in the teaching of mathematics requires teachers and researchers to think in depth about the relationship between the conceptual and the technical part of the mathematical activity rather than opposing them.

On a wider reflection Noss and Hoyles (1996, p. 21) stress the potential productivity of the constructivist approaches, but also their limits. First, when knowledge is built trough actions in a given computer context, pupils are able to produce powerful reflections on objects in this context to solve difficult problems, but it is not clear that this knowledge helps with tasks outside the computer context. It appears, therefore, very contextualised, and the decontextualisation is a problem. Second, the ‘procedure-object’ approach is sometimes a too rigid way for building concepts. There is no permanent necessity to consider first an ‘operative’ (Sfard, 1991) approach of concepts. In contrast, computers now offer a range of views (or windows) on a concept wider than just the procedure-object duality. For instance, the graphical utility is one between many views of the concept of function in a computer environment, and the resulting plot can be considered as a procedure (tracing the plot) and as an object (the global properties of the plot).

The computer’s role in the mediation of students’ activity

Noss and Hoyles (ibid., p. 54) point out the dialectic between human culture and technology. A ‘cognitive’ tool is made from human cognition and it has an effect on the cognitive functioning of a person who uses it. In this way, Noss and Hoyles stress, a computer application may operate as a linguistic tool, and they emphasise programming as a tool for expressing and articulating ideas. In their approach to the teaching of a topic like proportionality (p. 75), they combine paper and pencil problem solving, a computerised ‘target game’ and work on Logo procedures for drawing objects in proportion. The power of expression of the computer helps to broaden students' conceptions of multiplication and, working on Logo procedures, students act on the relation of proportionality and on a formalisation of this relation. In off-computer activity students are able to refer to the Logo formalism for explanation and evaluation.

So, in Noss and Hoyles‘ view, the computer environment is not only a field for students’ purposeful actions. The computer offers special means for interacting with objects. Using the means, students enlarge their conceptions of the objects, especially towards generalisation and formalisation. Therefore, Noss and Hoyles introduce this mediation as a major role for the computer in the student’s process of abstraction. This idea of mediation of instruments in the mental sphere of human activity was initiated by Vygotskii. Primarily, the mediation is the use of properties of a given object to act on another for a given task[5]. The point is that mediation changes the nature of the action of human over objects. In the psychological

[5] « Mediation is a trick of the mind » Hegel quoted by Moro, Scheuwly (1997, p.2).

sphere, Vygotskii's assumption is that "language, (..) algebraic symbols, (..) and all possible signs and symbols" are instruments which change the mental activity[6].

This idea of mediation is useful in our project because a purely constructivist view of the use of computers is insufficient to analyse the interaction between the user, his/her instrument and the objects in the settings. A constructivist view assumes that the computer settings will provide the means for a predictable and meaningful interaction. What actually happened when we observed the use of DERIVE was different: interaction situations of the students and DERIVE were often less productive than teachers' expectation. Teachers generally expected that students would build mathematical meaning from DERIVE's feed-back. Students' reactions and reflections did not have this meaning because their perception of the feedback was influenced by the operation of the software (Lagrange, 1996). For instance 9th grade students with little familiarity with DERIVE, were asked to observe the result of the *Expand* command on the square of algebraic sums. The teacher expected that the students would concentrate on regularities in this expansion like, for instance, the relation of the number of terms in the sum and in the expansion. In contrast students reflected deeply on the order of the terms in the expansion, which is a regularity only linked to the software. Mediation accounts for this phenomenon because students perceived the mathematical settings through DERIVE, and being unaware of the properties of this instrument, they could not understand that the regularities that they found had no mathematical significance. In contrast, the teacher was an expert both in mathematics and in DERIVE, and did not mind this regularity[7].

How do contemporary instruments like computers and calculators fit with a theory of mediation? A computer, as considered by Noss and Hoyles, is an instrument in two dimensions: a physical object with a keyboard, a screen and so on, and an abstract operative language. Noss and Hoyles focus on the abstract dimension of the Logo language, and therefore meet Vygotskii's view of mental instruments. In the use of complex calculators that I intend to analyse, this view seems less effective, particularly in the phase where the user is learning new capabilities. In this phase, a user sees the internal capabilities through the features of the interface (for instance, with a TI-92[8], the different capabilities for solving are seen through various entries of the algebra menus). This perception of the calculator does not distinguish between the interface and the internal logic. This phase of learning is what I want to analyse because, in this phase, cognitive processes are likely to appear, involving both the calculator and mathematics.

For this reason, I prefer to consider a calculator as a complex instrument like those existing in the area of professional working (for instance a computerised system to pilot a process) rather than to reduce it to an addition of a neutral interface and an internal algebraic language. An advantage of this approach is that it is easier to think about the changing relation of the user and his/her calculator: in this relation, the user discovers together the characteristics of his/her calculator together with the mathematical underlying features.

The role of the instrumental schemes

The process of development of new uses of an instrument, and the associated cognitive changes, have been analysed by psychologists in terms of conceptualisation. In a 'study of

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[6] Quoted by Moro, Scheuwly (ibid, p. 3).

[7] Artigue (1995) named « pseudo-transparency » this phenomenon: in the mediation, the instrument is transparent for the teacher, but not for students.

[8] In the paper, TI-92 may be replaced by TI-89 or other complex calculator with the same symbolic, graphic and numeric capabilitites. See note 1.

thought in relation to instrumented activity', Verillon and Rabardel (1995)[9] stress that a human creation, an 'artefact', is not immediately an instrument. A human being who wants to use an artefact builds up his/her relation with the artefact in two directions: externally s/he develops uses of the artefact and internally, s/he builds cognitive structures to control these uses. After Piaget, Verillon and Rabardel describe these structures in terms of schemes, which are mental means that a person creates to assimilate a situation. When a person acts on settings trough an instrument his/her behaviour has a specific organisation. For that reason, the authors[10] introduces the notion of 'instrument utilisation schemes'. These utilisation schemes have the properties of adaptation and assimilation of the schemes and direct the uses of the instrument by the person. Being mental structures of a person, utilisation schemes are not given with the artefact. They are built in an 'instrumental genesis' which combines the development of uses and the adaptation of schemes: when developing the first uses, a person pilots the artefact through existing schemes, then this primitive experience is the occasion of an adaptation of the schemes, and the better adapted schemes are a basis for developing new uses, and so on. This genesis is both individual and social: a person builds his/her own mental structures, but, generally, an instrument is not used by only one person and therefore the process of adaptation takes place in a social context.

Schemes in calculus using a complex calculator

Verillon and Rabardel 's cognitive approach to instruments shares many aspects of Hoyles and Noss' view of the computer in mathematical activity: the instrument is not something neutral, it has an effect on the cognitive functioning of a person who uses it. The cognitive approach describes this effect as the development of specific schemes, and organises this development in a genesis. This approach was stimulating for our project of integrating 'computer-like' calculators because they are complex devices with a lot of capabilities, each of them implying many specific schemes that the user has to co-ordinate to achieve a given task. The idea of genesis is useful because our project took place over a year and we had to think through the development of the uses and the schemes, together with the progression of the mathematical topics.

As an example, Figure 1 displays various schemes, calculator oriented or not, algebraic, graphic or symbolic that a user of a TI-92 can use to search for the variations of a function like $\frac{x^2+x+0.01}{x}$. The schemes have several dimension of functionality: decisional, they organise and control the action ; pragmatic, they act on the settings ; interpretative, they help to understand the settings.

[9] See also, in French, Rabardel (1995, p. 37).

[10] See also Rabardel (ibid, p. 93).

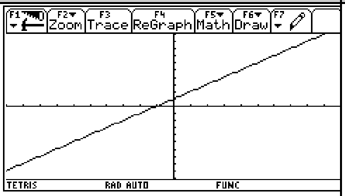
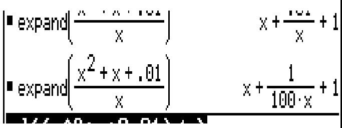
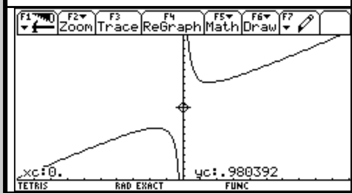
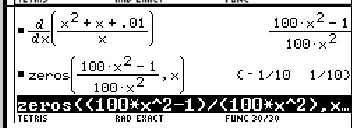
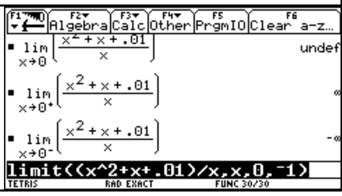
Nature of scheme	TI-92 output	Decisional dimension	Pragmatic dimension	Interpretative dimension
Graphic, TI-92		Graphing in the standard window is a good approach for the variations of a function	Consider the graph of the function in the standard window	Function is increasing. Graph is a straight line
Algebraic, criticism	none	Graphical evidence must be compared to algebraic aspects	Consider the algebraic definition of the function	$f(x)$ is not a linear function
Analytic, TI-92		Expanding an expression will give a linear and a fractional part, helping the interpretation of the graph	Consider another algebraic expression of the function	There is something special near x=0
Graphic, TI-92		Graphic display will confirm analytic ideas	Zoom in around x=0 and y=0 until something appears	There are two turning points near zero and there is a discontinuity
Calculus, TI-92		Use the derivative to search for the turning points	Find the zeros of the derivative	Position of the turning points is algebraically confirmed
Calculus, TI-92		When the result of limit is *undef*, then there are left and right hand limits	Adjust the *limit* command to obtain left and right hand limits	Nature of the discontinuity is found

Figure 1: Schemes in a search for the variations of a function

Being adaptive mental constructs, schemes cannot be entirely described in a rational form. In Figure 1, some of them are approached by their nature and by features of the above three dimensions. Many other schemes exist and are more difficult to describe. For instance, in the Graph window, a student often develop exploratory zooming based on his (her) private knowledge and previous experience. Moreover, schemes in Figure 1 are made of a number of 'sub-schemes' more difficult to explicit[11]. Nevertheless, the brief description of schemes in Figure 1 accounts for the complexity of an action with this complex instrument, and from this description, I will show what relation students may have with these schemes.

The first scheme (graphing in the standard window) is prevalent among most students. In the initial stages of learning calculus very few students are able to produce critical interpretations such as those in the second scheme, even when they have the algebraic knowledge to do so. The more able students develop schemes where graphical action is linked with algebraic and analytic interpretation: they see, in the graph, properties that they anticipate from an algebraic

[11] Trouche (1996, p. 303) produces a comprehensive analysis and classification of schemes in a search for a limit of a function.

analysis of the function. This co-operation of schemes of different nature gives them a new efficiency.

Transforming the expression of the function like in the third scheme is not a spontaneous action. Most students initially choose the transformation randomly among the TI-92 capabilities rather than from rational reflection. Teaching can help to develop this reflection. Switching back to the graph window, as in the fourth scheme, is quite natural. Some students anticipate immediately the required zooming, while others take considerable time over this decision. The latter may use trial and error processes, productive for some but unproductive for others.

The calculus approach in the fifth scheme may derive from a teaching method. I observed, however, that this scheme is activated only when the function is similar to standard functions considered in the teaching. When a student is perplexed, because of an unusual function, this scheme is not likely to appear. It may not appear with the example of Figure 1, because variations are not perceptible in a standard window. It certainly does not appear when a student meets a new type of function, for instance a trigonometric function when the student is used to rational functions.

The sixth scheme is about limits. It illustrates how specific an instrumental scheme may be. In ordinary paper and pencil practice, the notion of left and right hand limit is difficult because their computation implies a reflection on the sign of sub-expression which is not familiar to students. With the TI-92, the scheme described in Figure 1 works well on most functions and contributes to give sense to this notion. However, this sense is often partial, because most students have difficulties in interpreting the values of the limits in term of asymptotical behaviour of the graph.

In this brief description of features of schemes appearing in a calculus task, and their apprehension by students, the question of genesis appears with some complexity. The development of utilisation schemes by students appears to be linked to the development of their mathematical knowledge. But what is the nature of this link? Schemes appear to be more or less influenced by teaching. But what is this influence, and how is teaching to be oriented to help the development of suitable schemes, their generalisation and their co-ordination ? These questions call for theoretical and practical reflection that I will undertake in the following section.

An approach of teaching with instruments

Schemes for building knowledge

Rabardel and Vérillon's approach is an ergonomic one: finding a better way of conceptualising human-instrument relations. Hoyles and Noss' concern, as well as ours, is slightly different: to try to conceptualise how the use of instruments intervenes in the learning of mathematical topics. With respect to this aim we can go back to Vergnaud’s (1990) work on the role of schemes in conceptualisation: schemes organise the behaviour of a person in a class of problems and situations representative of a field of concepts and are a basis for knowledge in this field. A given concept, from this viewpoint, can be seen in relation to the set of problems to which it provides a means of solution, and knowledge of this concept derives from the schemes that a person builds to solve these problems.

When a person learns mathematics with an instrument, his (her) schemes organise behaviours related to the use of the instrument as well as more general conducts. Interpreting Noss and Hoyles' study of a teaching of proportionality, I can see that Logo programming is an instrumental practice for manipulating a formalisation of proportionality in a problem of expanding given patterns. Acting with this instrument, students develop utilisation schemes, for instance rules of transformation of Logo expressions to maintain the shape of a pattern. These schemes are specific and not directly transferred in a non-Logo context. But, together with other schemes, they are a frame for students' conceptual reflection, and they make specific contributions to that reflection.

At this point, comparing earlier approaches where mathematical knowledge is thought to be built from situations involving personal interaction with the computer, the potential contributions of computers and of calculators appears different: technology acts as a mediator for the action of students. In this mediation technology is by no means neutral: students have to elaborate utilisation schemes, a non-trivial task.

This approach is consistent with Hoyles and Noss' view of the role of technology in building mathematical meanings. In addition, I focus on the development of uses and utilisation schemes because in our project the students use a complex calculator over the course of a year as a everyday support to their mathematical practices. Given this, adequate utilisation schemes of 'hand held' technology are a condition for this support. In turn, the development of schemes (the instrumental genesis) is dependent on students' progressive understanding of the calculus. For this reason I emphasise this genesis and its role in students' learning.

The genesis is, however, problematic. Mathematical meaning and knowledge grow with the multiple schemes that students develop when doing tasks in a domain, but not all schemes are productive of adequate knowledge in all situations. Consider, for instance, the limit of a rational expression at a finite or infinite point. In an ordinary 'non computer' context, students may apply the following reasoning to, say, $\lim_{x\to\infty}\frac{1}{x}$: 'one over a large number will be small..., therefore, the limit is 0'. When the expression is more complex they may transform it, e.g. change $\lim_{x\to\infty}\left(x-x^2\right)$ into $\lim_{x\to\infty}\left(x(1-x)\right)$. Numerical and graphical approaches may contribute to students' progressive understanding of this task.

In contrast, with a calculator like the TI-92 or algebraic software like DERIVE, students are able to associate the idea of limit with a single scheme: pressing the 'limit' key of the calculator and reading the output on the screen. This scheme is effective for the task but, as Monaghan et al. (1994) observed, it may result in giving students a narrow understanding of the notion of limit. Comparing students who made extensive use of DERIVE with other students, they found that the latter had more varied representations of limits including infinitesimal approaches, whereas the DERIVE students focused solely on limits as objects. Viewing their report from the perspective of my theoretical framework, I say that the scheme associates too closely the idea of limit with the limit capabilities of DERIVE and this scheme generates a restricted mathematical meaning.

On the other hand, the scheme for right and left hand limits in the example of figure 1 is very close to the above 'key-stroke limit scheme'. I said above that it is productive when giving students a sense of the existence of the limits, otherwise hard to grasp, because of the difficulty of calculation.

So, depending on their co-operation with other schemes or meanings, schemes of use of the TI-92 or DERIVE are productive or not. Therefore, for the support of the technology to be effective teachers must control students' development of utilisation schemes and their co-ordination with the advancement of mathematical knowledge. However, there might be a contradiction here, because schemes are mental structures built by the student, rather than objects for the process of communication, like teaching. I thus examine the role of teaching in the context of the use of technology by students.

The role of tasks and techniques

I look at the teaching of techniques and at the relationship of this with the instrumental genesis, as this is a key point in the use of technology to teach and learn calculus. In Repo's (1994) research we saw above that approaches of this use may pretend to favour students' higher conceptual thinking, in opposition with the usual training to algorithms in the paper/pencil context. More precisely, authors and teachers assume that the symbolic capabilities in this technology are means to lessen the stress on techniques which, they consider, restrain students' reflection on concepts. This view was clearly present in teachers' expectations in the French DERIVE experiment, and reflecting on this was useful in establishing the limits of this excessively conceptual approach (Lagrange, 1996).

First, the technical work did not vanish when doing mathematics using Computer Algebra. Not all students welcomed the relief from the usual pen and paper skills: some of them considered these skills as important for success in Mathematics. It also appeared that using Computer Algebra itself required specific techniques. For instance, when a student obtains an output using the system, this output is not always the usual expression generally accepted in the pen and paper context. In this situation few students could transform the system's output to obtain the usual expression. A consequence is that although most students thought of Computer Algebra as a helpful tool for 'double checking', they generally lacked the techniques to perform effectively this double check.

Understanding mathematics with the help of Computer Algebra was not a view that students generally considered. Even when they enjoyed the new classroom situations they experienced using Computer Algebra, they generally did not recognise that these situations could bring a better comprehension of mathematical content because the situations focused on conceptual aspects of a subject, and not on the usual techniques associated with this content.

This observation was a starting point for a reflection on the relationship between the technical and conceptual part of mathematical activities. Chevallard (1992, 1996) stresses the links between techniques and theory. Every topic, mathematical or not, has a set of tasks and methods to perform these tasks. Newcomers in the topic see the tasks as problems. Progressively they acquire the means to achieve them and they become skilled. That is how they acquire techniques in a topic. Furthermore, in teaching and learning situations, the students and the teachers are not interested in simply acquiring and applying a set of techniques. They want to talk about them, and therefore they develop a specific language. Then, they can use this language to question the consistency and the limits of the techniques. In this way they reach a theoretical understanding of a topic.

A break from teaching based exclusively on training in algorithmic skills is certainly interesting. However, teachers' and researchers' views of the support of symbolic computation tends to hide the need for a set of techniques.

So, I emphasised above the role of schemes in the process of conceptualisation, and now I stress the need for techniques in the teaching of concepts. But what is the relationship between schemes and techniques? I said above that schemes, being internal adaptive constructions of a person, cannot be taught directly. In contrast, techniques are rational elaborations used in teaching. Techniques are official means of achieving a task but, in facing the task, a person doesn't 'follow' a technique, especially when the task is new or more complex or more problematic than usual. When knowledge is requested a person acts through schemes.

So, in an educational context, techniques can be seen as official, rational objects for communicating whereas schemes are structures actually produced in students' mind[12]. Drilling on a single technique for a given task without reflection is only able to produce manipulative schemes and poor knowledge. Many innovators, particularly in the field of the use of computer, argue this to diminish the role of techniques and try to promote 'conceptual mathematics'. I observed in the DERIVE experiment that diminishing the role of techniques encouraged teachers to avoid devoting time for discussion on these. In contrast, talking of techniques in the classroom might help students to develop suitable schemes. Furthermore, in this communication a specific language and theoretical reflection is able to appear and students can enhance the reflective part of their schemes.

Techniques in the use of a complex calculator

Returning the use of symbolic computation, graphical, numerical and symbolic facilities make traditional techniques less relevant. In addition, the role of those techniques is often undervalued because teachers see them as the routine part of their activity. New techniques should be taught to help the development of utilisation schemes but teachers often believe that these techniques are obvious or linked too closely to the calculator to be relevant. The necessity and relevance of new techniques may be made clear by considering the task in Figure 2. It was given in a French experimental exam designed to test the adequacy of a set of questions when students are allowed to use calculators. The text is written to avoid giving advantage to students with a symbolic calculator: a factorised expression of the derivative is given, so students without symbolic facilities are able to do the subsequent question (variations of *f*) in a similar manner to students who obtain this expression from their symbolic calculator. But, when I consider the TI-92 answer for the derivative, I see that the task of the user with a symbolic calculator is not straightforward. The TI-92 answer is neither the expression of the text nor the raw form obtained when applying the rules of differentiation. Recognising the expression of the text as a factorised form the user may apply the *factor* command. Again, the expression is not the same as in the text. Therefore the user has first to show how the TI-92 answer can be obtained from the raw differentiation and then reflect on the two TI-92 expressions to show their equivalence with the expression of the text. Techniques exist to do that (for instance, differentiating sub expressions helps to obtain the raw form, reflecting on the desired form helps to choose the right application), and, although linked to the calculator, these techniques might be a topic for teaching. For instance, reflecting on the desired expression on theTI-92 may help students to focus on the forms of the expressions.

[12] In this paragraph, I look briefly at the relationship between schemes and techniques, to emphasise their respective functions. Schemes and techniques may be viewed in a more dialectical relation. There is a wide range, from personal hidden elementary schemes to social global schemes. The latter are more easily rationalised. Teaching can act more directly on these schemes, very similar to techniques. See again Trouche (ibid.)

The text of the question

Consider the function *f* define for strictly positive real numbers by

$f(x) = x\ln(x) - 2\ln(x) - (\ln(x))^2$.

Demonstrate that the derivative of *f* is given by $f'(x) = \left(1 - \frac{2}{x}\right)(1 + \ln(x))$

Using the TI-92 to answer

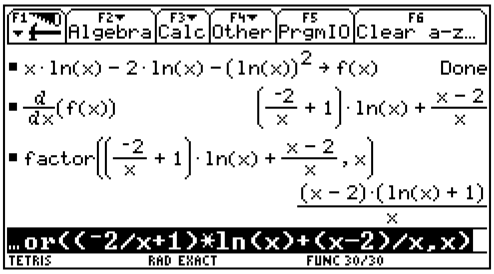


Figure 2: a task in an experimental exam

TI-92 techniques are specific because they rationalise schemes of use of an instrument, and, according to Rabardel and Vérillon, these schemes develop in an instrumental genesis. A consequence is that the organisation of the tasks and associated techniques must comply with the constraints of that genesis and direct it in a productive way: schemes cannot develop arbitrary and not all combinations of schemes are productive for mathematical meaning. Below I look more closely at these constraints and their implications in terms of tasks and techniques, from the experience of the TI-92 project.

Teaching pre-calculus with computer-like calculators

Our team choose a level where the legitimacy of an unusual and relatively expensive calculator might be accepted[13]. In the French general upper secondary level, students are in three main branches: literature, economy and science. In this latter branch students' use of calculators with sophisticated numerical and graphical capabilities is now well established. Thus, we expected that students accept the TI-92 in spite of its unusual aspect, as an 'enhanced' substitute for their familiar calculator. We chose the first year (eleventh grade), because the 'baccalaureat', at the end of the second and last year of this course brings students much anxiety, with possible negative effects on the experiment. The curriculum of this first year is an introduction to calculus concepts (functions, limits and derivatives) and to their application, based on problem solving and on experimenting. This curriculum suited our approach well, because it focuses on the development of abilities in algebra and calculus and on the understanding of functional concepts, an interesting frame for an instrumental genesis.

We worked with two teachers in two distinct regions of France. Thus, although the teachers collaborated, we might observe two distinct experiences of the integration of the TI-92. In the first year, our work was mainly observing classroom sessions and students. The teachers had been working with us in the DERIVE experiment, and we asked them to adapt the many sessions that teachers built in this experiment, in order to use the calculator in every suitable classroom situation.

The observation of the students was done by way of three attitudinal questionnaires and three individual interviews of a sample of students. From the observation in the first year and from

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[13]See footnote 3.

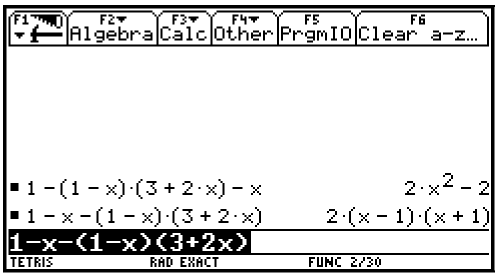
the analysis of classroom sessions in the same year, we built our project, a series of lessons and classroom activities that the French Ministry of Education will publish as a guideline for teachers. We experimented this project in the second year: the teachers taught the lessons and we did an observation like in the first year.

The aim of this paper is not to report this whole experimentation, but to emphasise the role of teaching. Lagrange (to appear) will focus on the observation of students. It will show how, in the first year, the acquisition of utilisation schemes was a long and complex process, effective for some students and more problematic for others, with significant differences between individual students and between the two classes. It will also discuss the improvement that the project that we experimented the second year brought in students' attitudes and abilities. Here, in this paper, from the lessons that we experimented in the second year, I offer a view on how teaching might help the development of schemes productive to mathematical meaning. The observation of students' genesis in the first year will be used to show the necessity of this view[14], and classroom observations in the second year will help to discuss its effectiveness.

Tasks and techniques to develop an appropriate instrumental genesis for algebra and functions

Obviously, at the beginning of an instrumental genesis, a user exercises the schemes s/he built for other familiar instruments. For instance, when a beginner uses his/her new TI-92 to do a division, like 34 divided into 14, s/he keys in $\boxed{3}$ $\boxed{4}$ $\boxed{/}$ $\boxed{1}$ $\boxed{4}$ $\boxed{=}$ like on an ordinary numerical calculator and s/he is very surprised when the TI-92 answers 17/7. Tasks and techniques are to be organised to help him/her learn that, in the default mode, the TI-92 simplifies radicals and rational numbers symbolically and that decimal approximations must be specifically requested. Moreover, the user has to consider that the graph window handles functions in an approximate mode. Meanwhile, s/he has to consider, more acutely than usual, the difference between the mathematical treatment of numbers and the approximations of everyday practice.

Then schemes of use of the algebraic capabilities are essential. Symbolic applications like DERIVE or the main module of the TI-92 are basically algebraic, even when they include facilities in calculus: their core is the treatment of expressions that a student has to understand before s/he is able to use them for problems on functions. A key point is the notion of equivalence of expressions and the need for awareness of the different equivalent forms of an expression. A TI-92 user meets automatic simplification as soon as an expression is entered. The following screen displays an example of a puzzling phenomenon occurring with an automatic simplification. Two obviously equivalent expressions are ‘simplified’ in two radically different forms.

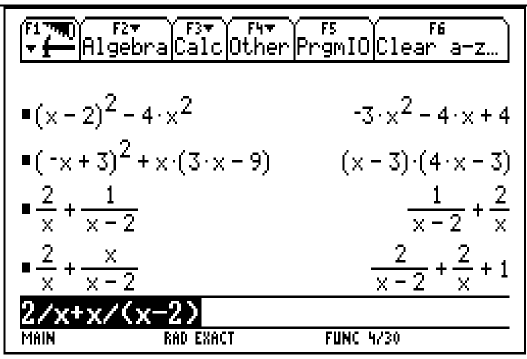


So a student cannot rely on automatic simplification to obtain the form s/he needs for an expression. S/he must consciously learn to use the items of the algebra menu (*Factor*,

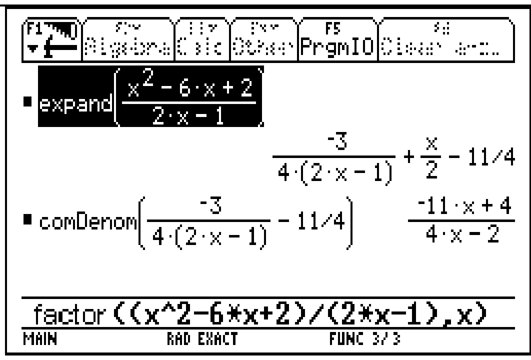
[14] Defouad (to appear) analyses more comprehensively the varied individual genesis of students.

Expand, *ComDenom*), to decide whether expressions are equivalent as well as anticipate the output of a given transformation on a given expression.

The following screens (Figure 3) give a hint of the tasks involved in our scheme, to develop students' algebraic instrumental schemes.



Screen A



Screen B

Figure 3: algebraic tasks

In a first task students had to enter the expressions on the left of the screen A and then observe the TI-92 simplification on the right of the screen A. They then identified the mathematical treatment: expanding, factoring, ordering, partial fractional expanding. In another task, an expression G was given, with three other apparently similar expressions H, I, J. The student had to find a TI-92 command to decide what expression H, I, J is equivalent to G. In this task, G was $\frac{x^2-6x+2}{2x-1}$, H was $\frac{-11x+4}{2x-1}+\frac{x}{2}$, I was $\frac{3}{4(2x-1)}-\frac{x}{2}+\frac{11}{4}$ and J was $\frac{\left(x+\sqrt{7}-3\right)\left(x-\sqrt{7}-3\right)}{2x-1}$. By expanding G, students had no difficulty seeing that I was the opposite of G (screen B). In contrast, showing that G and H are equivalent, is not straightforward: in the screen B, the function for the reduction of a sum of rational expressions has been used to reduce a sub-expression. *Factor* was the appropriate function to obtain J from G.

Work on the equivalence of expressions proved necessary not only at the beginning of the TI-92 use. For example, towards the end of the academic year of student TI-92 use the teacher asked them to differentiate the trigonometric function $x \to \cos\left(3x-\frac{\pi}{6}\right)$ by hand and with the TI-92, and to explain why resulting expressions are equivalent. The application of rules of differentiation gives $-3\sin\left(3x-\frac{\pi}{6}\right)$ when the TI-92 gives $3\cos\left(3x+\frac{\pi}{3}\right)$. We expected that students could give a reason like $\cos\left(a+\frac{\pi}{3}\right)=-\sin\left(a-\frac{\pi}{6}\right)$, because they knew the property $\cos\left(a+\frac{\pi}{2}\right)=-\sin(a)$ but only 8 in a class of 26 were able to do that. Others expressed general reasons like "the calculator doesn't work like we do". So the work on the equivalence of expressions had to be continued when new expressions were introduced to help students to build suitable utilisation schemes.

As stated before, conscious use of the algebraic capabilities of the TI-92 may help students to focus on the most suitable form for a given task, whereas paper and pencil schemes focus on the rules of transformation. One may reasonably think that the joint development of the TI-92 and paper/pencil schemes is able to give an understanding of the equivalence of expression. This is an example of how paper and pencil and TI-92 practices are to be thought complementary in teaching, rather than opposed.

Like many calculators, the TI-92 offers a graphical window and a numerical table with a wide range of capabilities. Therefore, it may enhance early functional thinking because graphical and numerical schemes are essential for the growth of the function concept. As seen above, in Figure 1, notions like the variations of a function implies the co-ordination of algebraic and graphico-numerical utilisation schemes. Able users adjust settings of the graph window to make visible properties that they see algebraically, and use algebraic transformations to prove properties that they read from a graph.

A relevant task for developing these schemes is the study of functions whose properties are not obvious in a standard graph (see Figure 1 for an example, and Guin, Trouche, 1998 for others). From a task like this, teaching may focus on techniques for useful zooming (identifying values of interest, specifying the graph window to show those values...) and for relevant algebraic transformation (*Expand* for finding the asymptotic behaviour, *Solve* for the intersection with axis...). From this, students can get a better view of the treatment of functions in the graph window.

Helping students to develop flexible links between calculus concept representations

Work and schemes in algebra and functions is not itself calculus, but it is the first part of a genesis in which limits and differentiation can develop. The use of the TI-92 in pre-calculus is quite simple: a menu entry for the symbolic calculation of limits and a key for the calculation of derivatives. So, as a difference with algebra and functions, no specific instrumental learning will be necessary. However, we saw above, with limits, that this use tends to produce symbolic manipulative schemes, likely to generate a narrow understanding of these concepts if they are alone. This implies a deep reflection on how teaching can help the development of other schemes, instrumental or not, that students could associate to the idea of limit and derivative.

Tall (1996) stresses that there is not a single way, but ‘a spectrum of possible approaches ... from real-word calculus ... through the numeric, symbolic and graphic representations in elementary calculus, and on the to the formal ... approach of analysis’. He emphasises the need for helping students to move flexibly from one representation to another. He notes, moreover, that in teaching the balance between various approaches is difficult to establish because an approach well suited for one student will not necessary suit another, due to differing cognitive profiles. Traditional balances exist but technology tends to toss them about.

For instance, in France, every student in the secondary level now has a graphico-numerical calculator. This situation clearly changes the balance of numeric and graphical representations and of the symbolic view of the concepts of calculus. For instance, in a traditional approach, searching for the variations of a function began with symbolic study. The result of this study was a set of particular values of the variable (turning points, asymptotic branches...). Values of the function were calculated for those particular values: they were few because of the lengthy calculations by hand. These values helped to scale, then plot the graph of the function, which, in turn, helped to check the consistency of the symbolic study. So, in this traditional approach, the symbolic study commanded the study of variations. In contrast, with numericographical calculators, graphing, zooming, calculating tables of values is very easy and a student will use these facilities extensively to get graphical or numerical evidence of limits or variations. Symbolic calculations are relatively difficult, as they necessitate paper and pencil work, and a different reflection. Many authors emphasise the potential of graphic calculators for developing new understandings and skills (see for instance Shoaf-Grubbs, 1995). It is, nevertheless, true that these calculators make the traditional balance of symbolic and graphico-numerical representations redundant, but a new balance is not yet clearly established. A cautious approach would be to encourage students to use a symbolic approach to control their calculator work, but students often prefer experimenting rather than analysing. Trouche (1996) noticed this behaviour and emphasises the schemes that students should develop to control the graphs and numbers they obtain on their calculators: these schemes are theoretical (through algebraic or analytic considerations), enactive (the student acts on the picture, for instance s/he traces a graph) or reflective (producing other graphs/pictures). He suggests specific teaching strategies to help students to get these schemes.

Now, with the TI-92[15], calculators are graphico-numeric *and* symbolic. Little is known of how this new feature will affect the balance between representations. In most studies, students do not have enough practice of computer algebra to effect a change in the links between representations, or authors do not consider this question[16].

A first difficulty is the higher complexity of this instrument. With common graphing calculators, graphical and numerical schemes are instrumental when analytic schemes are associated with pencil and paper practices. In contrast, with the TI-92, co-ordinating analytic and graphico-numerical schemes implies controlled switches between windows. In those switches, a flexible view of the organisation of functions in the calculator is essential. An example of the difficulties occurring because of a limited view was observed with one student. She had limited knowledge of the capabilities of her TI-92 and one of her peers entered the following into her calculator:

So, the second function, y2, was systematically defined as the derivative of the first function y1. When she had to study a function, she introduced this function as y1, and then, without doing anything, y2 was its derivative. She did know that, but not the way it worked. So in ordinary tasks with her calculator she was comfortable. But when she had to use another calculator, or in a task involving two functions, she was totally confused.

Whatever the difficulties, the design of teaching modules for pre-calculus courses forced our team to make assumptions on how, with easier symbolic calculation, teaching may obtain an adequate balance between representations. We assumed first that easier symbolic calculation enlarges the possibility of linking enactive representations and theoretical calculus, and also that teaching must avoid the danger of too close an association between concepts and symbolic manipulative schemes, wiping out other representations.

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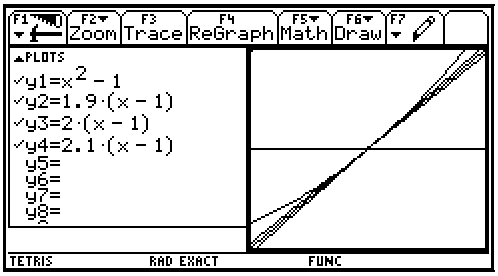
[15] and other calculators: see note 1.

[16] Ruthven (1997) reviewed a number of researches into CAS in mathematics education. The prevalent topic appearing in this review was the comparison of student performances between CAS and non CAS students. Research reports on the impact of CAS in the everyday teaching are very recently available. One of them is Guin, Trouche (1998)'s study.

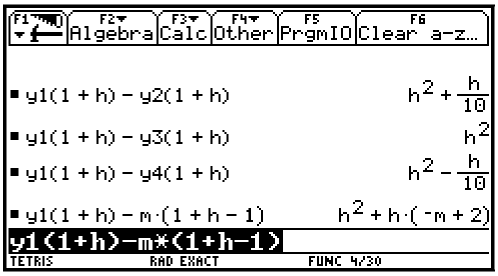
Linking enactive representations and theoretical calculus

Enactive representations (Tall, 1996) exist in the prior differential knowledge of students. For instance, most students have a sense of the tangential behaviour of curves from their geometrical experience. It seems important to use this knowledge as a basis for the theoretical concept of derivative, because differentiation is an analytic answer to the question of the tangent line for a curve defined by a function. However, in the ordinary context of paper and pencil calculations, students cannot really question their enactive differential notions because they would have to consider, and give sense to, expressions which are beyond their experience[17]. Using symbolic computation potentially helps students to work with these expressions and to understand their meaning.

As an example, let us consider the introduction of the concept of derivative we experimented. It starts from the following problem: Let (G) be the graph of the function *f* defined by $f(x) = x^2 - 1$and A be the point (1 , 0). For every straight line passing through A, a number *m* exists such that an equation is $y = m(x-1)$. What straight line gives the best fit of the graph (G)? Through geometrical experience students were able to guess that the line of best fit is when *m*=2. But the teacher stressed that, for *m*=1.9 and *m*=2.1, the lines also fit well, and students recognised that zooming doesn't help to distinguish the 'fitness' of the three lines.



From this reflection it appeared necessary to consider the distances between an arbitrary point on the curve, near A, and the points of same abscissa (1+*h*) on the lines. The first three lines of the TI-92 main screen below display the expressions of this distance for the three lines.



The simplified forms (on the right of the screen) gave a hint of why the line fits better for *m*=2. In ordinary paper and pencil practice, the expressions and simplification would have been complex for students and would have hidden the sense of the expressions. A feature of symbolic computation is that students were able to focus on the symbolic forms without being

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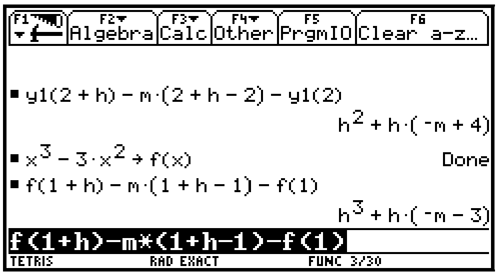
[17] Motion and velocity are other enactive differential notions that could be considered. Questioning this notion seemed even more difficult for students. So we did not consider this notion in the introduction of the concept of derivative. It seemed however interesting that students establish the link between this notion and the differentiation when the concept of derivative and associated schemes were steady enough.

For a very stimulating picture of problems arising when students have to build mathematical representations of motion, see Boyd and Rubin (1996). Interestingly, they study the effect of mediation by a non-computer technology: the interactive video.

disturbed by the complexity of hand calculations[18]. Students were, moreover, able to address the more general question of what line fits the best among all lines passing through A. Calculating with a parameter is never easy for students at this level, but the symbolic computation (last line of the screen) made it similar to the preceding calculations.

Furthermore, given the aim to develop students' links between their enactive conceptions of tangent lines and the theoretical notion of derivative, one example is clearly not enough. Symbolic computation again may make a contribution because students were able to address the same question, first for other points of the same graph (G), then for other functions. The example below shows a screen for the same function defined by $f(x) = x^2 - 1$ at the point x=2, then for a cubic function at the point x=1.



In this process, students operated by themselves progressive modifications of the above expression, and in doing so, concentrated on the expression and grasped its sense[19]. The symbolic capabilities of the TI-92 were essential in helping students to focus on the algebraic forms. However, they needed good algebraic schemes of use of the TI-92 to give sense to the transformations.

Symbolic aspects of calculus concepts

I expressed my concerns above that students may use the symbolic capabilities for very simple limits or derivatives and see nothing more in those concepts than the manipulative aspects. In the paper and pencil context the tendency for students to consider mathematics as meaningless symbolic manipulation exists, but these manipulations are often tedious and better replaced by a reflection which involves other representations of the concepts. As Monaghan et al. (1994) argue, symbolic computation may make manipulations effortless but tends to obscure other representations linked with infinitesimal approaches. For that reason, we preferred to introduce the TI-92 capabilities for limits and derivative only after students did considerable work on the concepts, linking enactive views with graphico-numerical approaches and symbolic forms. This is clearly distinct from approaches like Watkins (1992) which introduce calculus concepts as outputs of symbolic computations. Watkins' concern, however, is with vocational courses where little time is devoted to conceptual developments. At the secondary general level the balance of representations should be a major goal.

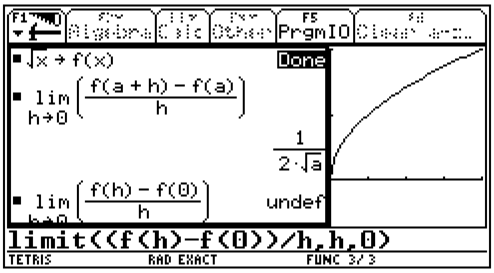
So, in our proposition, the limit concept was introduced from an intuitive view that a function 'tends toward zero as x tends toward zero'. Students did a lot of graphic and numeric work, passing from ' $f(x)$ is small when x is small' to '$f(x)$ can be arbitrarily small provided that x is

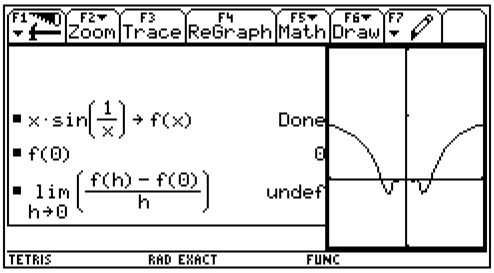
[18] Students were however interested for knowing the reason why the expressions simplified in that form. That is why the teacher asked them to develop and simplify with paper and pencil one of the expressions. Done separately of the main process of solving, and for only one expression, this hand calculation gave this process a complementary meaning without obscuring it.

[19] This process has clear links with the formalisation of proportionality in expanding patterns of Noss and Hoyles: pupils have a perceptive (enactive) idea of patterns in proportion, and the computer helps them to consider a formalised relation between those patterns.

small enough’. Then students had to study by the same means standard, as well as non regular, limits before the *limit* function of the TI-92 was introduced.

The concept of derivative was introduced from a geometrical representation, as outlined above, and another two sessions focused on activities where students had first to build a formal definition of the derivative and then as for the limits, search for derivatives of standard as well as non regular functions. In this work, students used the symbolic and graphic facilities of the TI-92, but not the key for the symbolic differentiation (see screens).





After this it was time to consider the symbolic aspects of the concepts, namely the algebraic rules by which a person or a machine is able to obtain derivatives or limits of expressions. The question was how to use the TI-92 to teach students algebraic rules that their calculator uses, and to give those rules a meaning.

Authors introduced the “black-box-white-box process” which could be an answer to the above question.

> *Using the CAS*[20] *as a black box enables students to discover mathematical theories, concepts or algorithms (...) to the point where the students say ‘we are able to do what the CAS can do’* (Heugl, 1997, p. 34).

This is, in my opinion, too simplistic a view of the support computer algebra can give: using a CAS as a black box, students will only discover symbolic entities. Learning theories and concepts implies wider strategies, as we have seen above. However, as the black-box-white-box process focuses on symbolic aspects of concepts, it could be useful for teaching symbolic rules.

For instance, students could consider several examples of how the TI-92 computes limits and derivatives and then learn to do those calculations by themselves. In this process the student is likely to have more self-reflection than in a formal approach where the teacher demonstrates the rules. However, from experience, we consider that implementing such processes is not so simple. The first problem is that in this process students are prone to see only the manipulative aspect of the rules, even when previous teaching focused on other representations of the concepts.

Another difficulty occurs when students have insufficient algebraic maturity to give a suitable meaning to the feedback from the computer. Pozzi (1994) gives an example where students had to understand the rule for the differentiation of a product of functions. They considered the differentiation of $x \to \cos(x)\left(7x^3 + 2x\right)$, and, using DERIVE, they obtained $(21x^2 + 2)\cos(x) - x\left(7x^2 + 2\right)\sin x$. Then, they deeply reflected on the central part of the expression $\cos(x) - x\left(7x^2 + 2\right)$ and found it to be very similar to the original function. They tried unsuccessfully to derive a general rule from this example. Clearly, their reflection was

[20] Computer Algebra System

misplaced because they did not see that the central part is not a sub-expression of the derivative. Once more, we see how good algebraic schemes are essential to be able to make sense of computer algebra output.

Pozzi further stresses that ‘computer algebra systems *can* support students to make sense of their algebraic generalisation’ but he maintains that ‘this is only likely to be achieved if (students) use the computer to explore and verify their conclusions and not simply as a symbolic calculator’. So students should be encouraged to make conjectures about general properties and produce examples to test these conjectures.

As an example of this we designed a session to enable students to discover the algebraic properties of limits and to learn how to use them.

This algebra of limits is summarised in the following TI-92 table.

I	L	I+L	IL|k>0	I/L
c1	c2	c3	c4	c5
-∞	∞	undef	-∞	undef
k	∞	∞	∞	0
k	-∞	-∞	-∞	0
0	∞	∞	undef	0
0	0	0	0	undef

In this table, the four indefinite limits appear ‘*undef*’. The actual aim of the lesson was that the students bear in mind these four cases as well discovering the means to solve these limits. For instance, they should be able to recognise that $\lim\limits_{x\to 0}\left(\frac{1}{x^4}-\frac{1}{x^2}\right)$ is indefinite and to find that this expression has actually a value ($+\infty$).

We asked students to experiment on an example of explicit functions and not on symbolic notations like in the above table. The data-matrix editor[21] of the TI-92 was used to support this investigation. The teacher introduced the first examples of limits of sums to give students a method, and to introduce the problem of indefinite cases. Students were then requested to produce others examples of possible values for $\lim\limits_{x\to 0}\left(f(x)+g(x)\right)$, when $\lim\limits_{x\to 0} f(x)=+\infty$ and $\lim\limits_{x\to 0} g(x)=-\infty$. Then they had to make conjectures for products and quotients and produce examples to illustrate these. The TI-92 gave the values of the limits but the teacher asked the students to explain the values by qualitative reasons or by calculation.

At this time students’ knowledge about limits was new and they encountered many difficulties recalling even simple limits. We were aware that this lack of mathematical maturity might cause them too much to rely on the TI-92 for the calculation of limits and thus to use try and error strategies rather than anticipating. After a first experimentation of the lesson we decided to make the limit point zero for all the limits.

With these settings students produced a lot of examples and convinced themselves that if $\lim\limits_{x\to 0} f(x)=+\infty$ and $\lim\limits_{x\to 0} g(x)=-\infty$, $\lim\limits_{x\to 0}\left(f(x)+g(x)\right)$ may ‘give everything’. The other indefinite cases did not appear immediately. For example, $\lim\limits_{x\to 0}\left(f(x)\times g(x)\right)$ presented difficulties when $\lim\limits_{x\to 0} f(x)=+\infty$ and $\lim\limits_{x\to 0} g(x)=0$, because many students produced examples where this limit was zero, and thought it was a general rule. But other students

[21] This module is like a symbolic simplified spread-sheet software.

produced examples where this limit was different, and convinced the class that it is again an indefinite case.

The need for controlled anticipation induced students to think of limits on the basis of their prior infinitesimal knowledge and the emphasis of the teacher that their reasoning corroborate values obtained of the TI-92 helped here. The problem of indefinite cases clearly appeared to students, and they could easily recall them from the examples that they produced. This is an example on how the symbolic calculator might help the students to conduct a mathematical activity in symbolic aspects of a concept in calculus without forgetting previously constructed representations.

A number of sessions of more open research followed these introductory lessons. This paper is a first look into how a teaching of pre-calculus might help the development of productive calculator use schemes. Thus, it leaves the students at a first stage of their genesis. Analysis of the sessions would help to see how in these sessions students put their schemes at work, questioned and enhanced them.

Conclusion

To conclude, I will outline the issues discussed in this paper, and point out questions for further analysis.

The table in Figure 4 summarises the key points which arise when I tried to conceptualise changes in the mathematical activity in a classroom when every student uses a 'computer-like' calculator and to see how teaching can take this use into account in a subject like pre-calculus. I saw the role of mediation of these calculators from the many new potentialities and constraints that they bring : when a student has one of these for everyday work in mathematics, his/her action depends strongly on these. Using it along a year s/he develops schemes specific to the calculator, together with other schemes. This instrumental genesis has its own constraints deriving from the specificity of the calculator as well as of the mathematical topic. As a student understands a mathematical topic from the schemes s/he builds to do tasks in this topic, teaching has to be attentive to this genesis. The teaching experiment I did with the DIDIREM team is a practical example of how a reflection on the instrumental genesis helps to design lessons, developing students' suitable schemes and connecting various representations of concepts.

The role of schemes in the understanding of mathematics is not a new idea. In contrast, the need for conceptualising the development of specific schemes of use recently appeared in the research studies when students uses of technology moved from occasional to regular. The context of long term everyday use of technology forced researchers to look at this instrumental genesis. Techniques are now seen by mathematics educators as an important level between the tasks and the theoretical reflection. However, this role has rarely been considered in the use of computerised tools. This paper offers to look at the techniques as official, rational objects in the classroom and to schemes as more 'private' entities making up a frame for the learners' knowledge. Highlighting various techniques and encouraging discussion on them, teaching influences students' development of utilisation schemes and is thus able to direct it in a mathematically productive way.

In this paper, my approach of the changes induced by a complex calculator in the learner's action was a broad one and issues would deserve further analysis. Particularly, I had just a short look at the effect of calculator language use on the students' work. A reasonable assumption would be that this language gives students an 'expressing power' that they could use when working with the calculator, and also in classroom interactions as observed by Hoyles and Noss (ibid. p. 153). These authors however demonstrated that this potential is not a general property of the use of technology, but a consequence of particularities of the microworlds that they analysed. Thus a more precise analysis of students' uses of calculator expressions to handle objects is to be done. This analyse should search for the possible schemes and technologies which would give sense to these uses. More generally, with or without calculator, we have to consider the instrumental dimension in students' work. A deeper look into this dimension would help to appreciate the respective contributions of the paper/pencil work and of calculator use.

	Mediation	Learning pre-calculus
Situations of use of a complex calculator	• new potentialities • new constraints	• specific utilisation schemes (analytic, graphical, symbolic)
Instrumental genesis	• productivity of schemes • role of techniques	• tasks to develop algebraic and functional schemes • situations to link representations in calculus (enactive-theoretic-symbolic)

Figure 4

Key points in teaching pre-calculus with complex calculators

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