Mathematical Working Spaces: designing and evaluating modelling based teaching/learning situations

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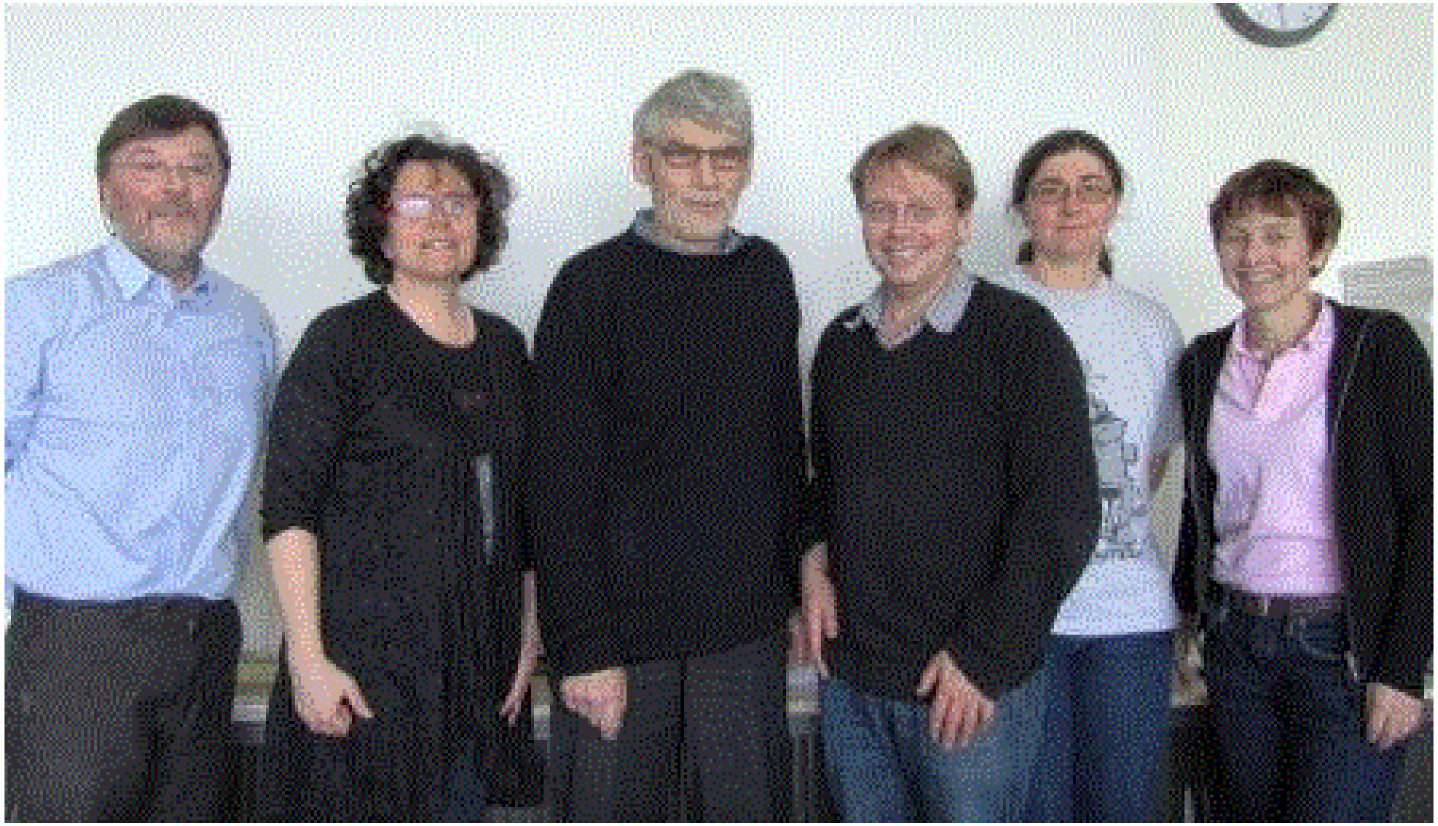
Working with teachers




My home in Greece

Preparing
the
seminar
☺☼☺

DEVELOPING THE TEACHING OF FUNCTION AND CALCULUS IN UPPER SECONDARY EDUCATION (TeaFuCal)

The teaching of calculus in schools (Torner, Potari & Zachariades, 2014)

- far away from what the students can understand.
- reduced to algorithms for algebraic calculations
- teachers face difficulties to promote deeper understanding
  - even when innovative materials are available.

Different approaches to facilitate understanding

- attempting to a smooth transition from informal to formal knowledge
- use of contextual tasks, modelling
- use of interconnected digital representations (Lagrange, 2014).

The aim is to improve the teaching of function and calculus in the upper secondary schools ...

Outline

- Modelling
- Working spaces
- Question:
  - How the idea of connected working spaces can help to
    - design
    - and evaluate

  modelling based teaching/learning situations

The modelling cycle

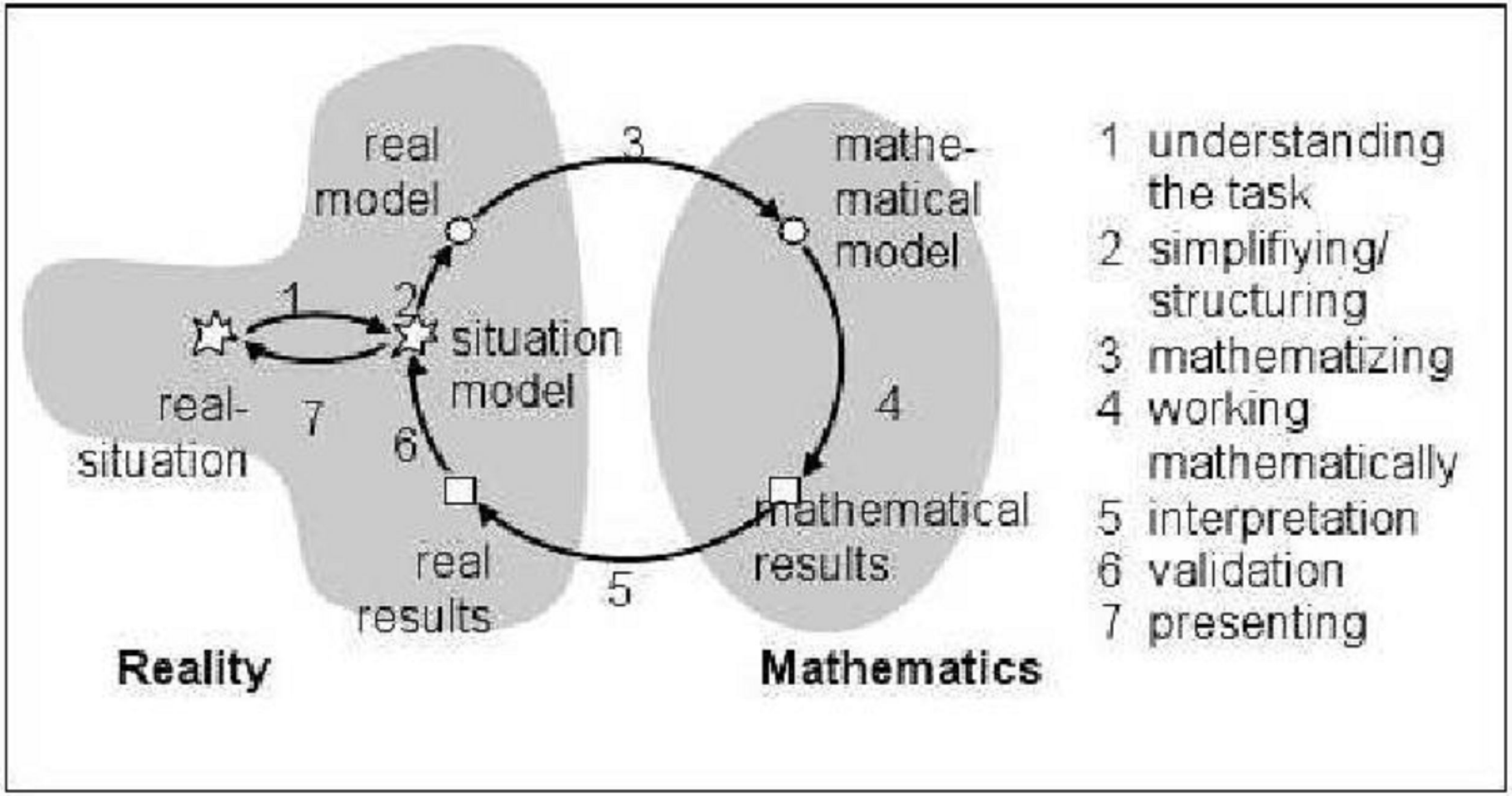
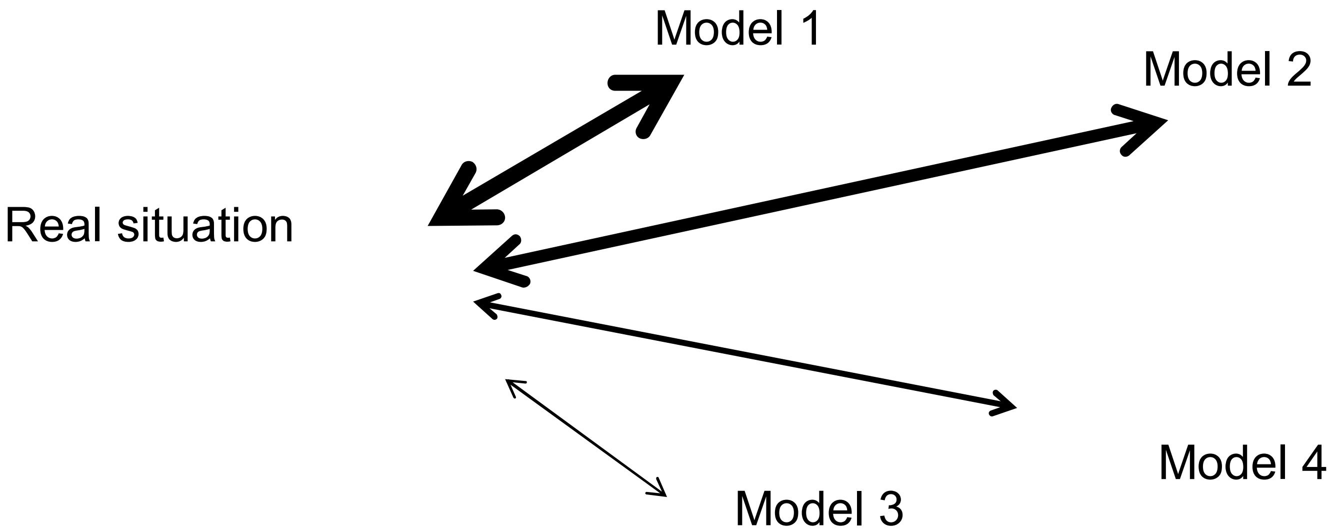
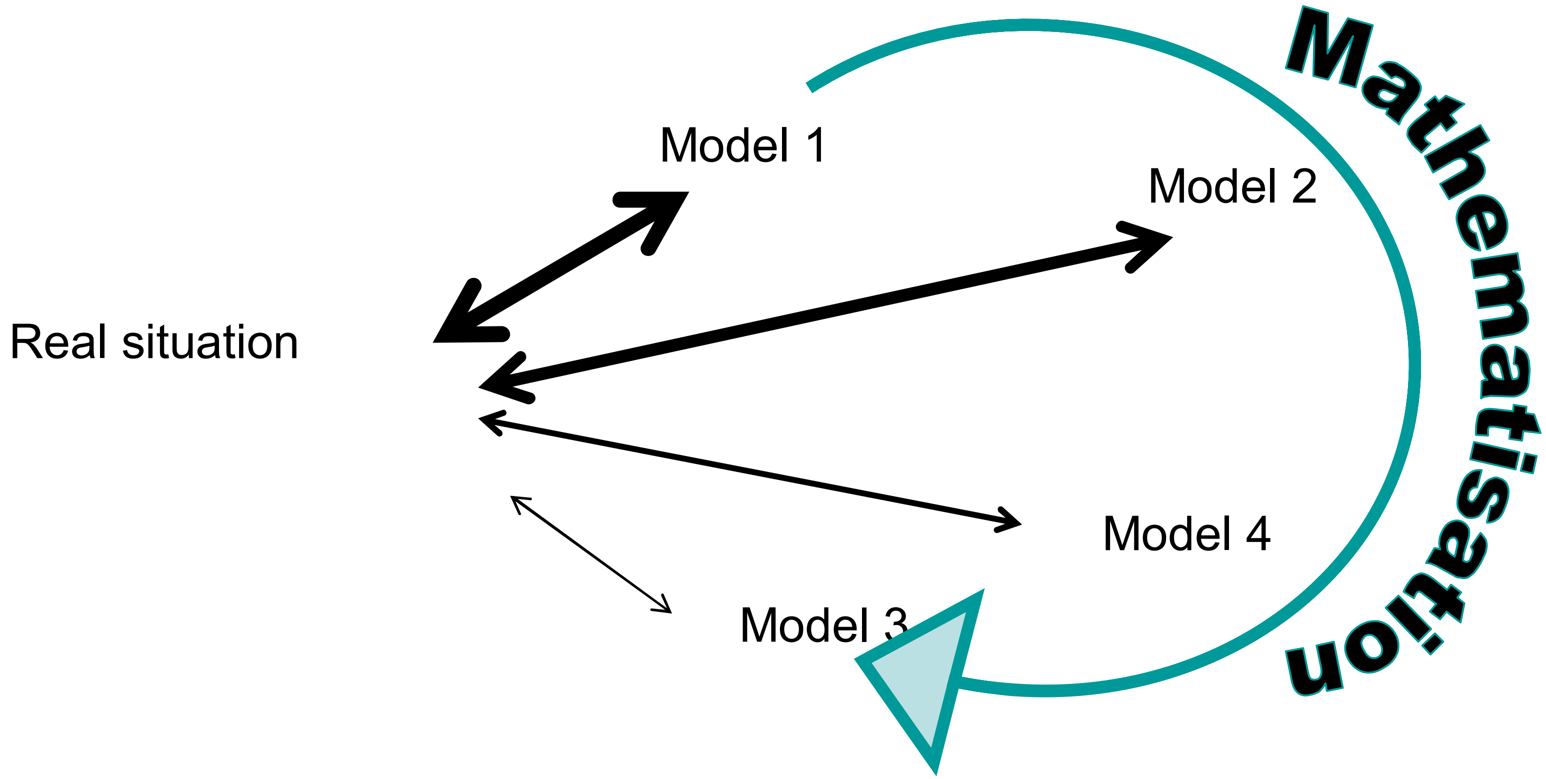


Figure 2 Le cycle de modélisation de Blum et Leiss (2005)

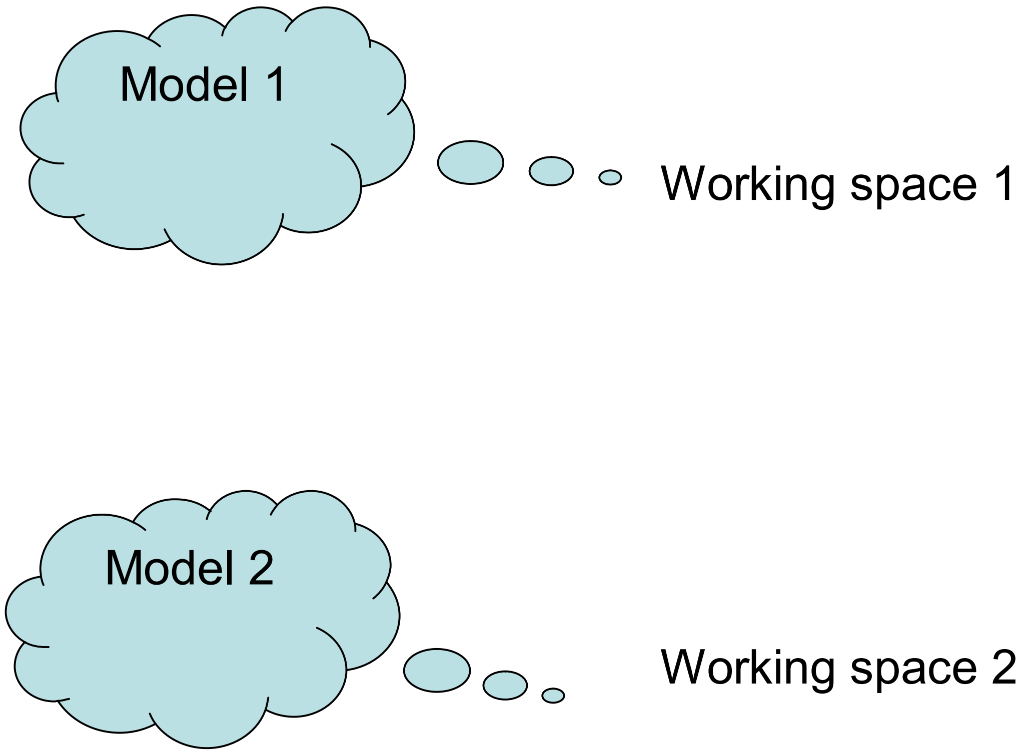
Different models of a situation



All models are mathematical, some are more



Working on each model is working in a specific space



An example

- Alfonso has just come from a journey in the pre-cordillera where he saw a field with a quadrilateral shape which has interested his family.
- He wants to estimate its area.
- For that, during the journey, he measured, successively, the four sides of the field and he found, approximatively : 300 m , 442 m, 608 m, 916 m,.
- Advised by his friend Rayen, he also measured a diagonal :632m.
- Could you help Alfonso to determine the area of the field?

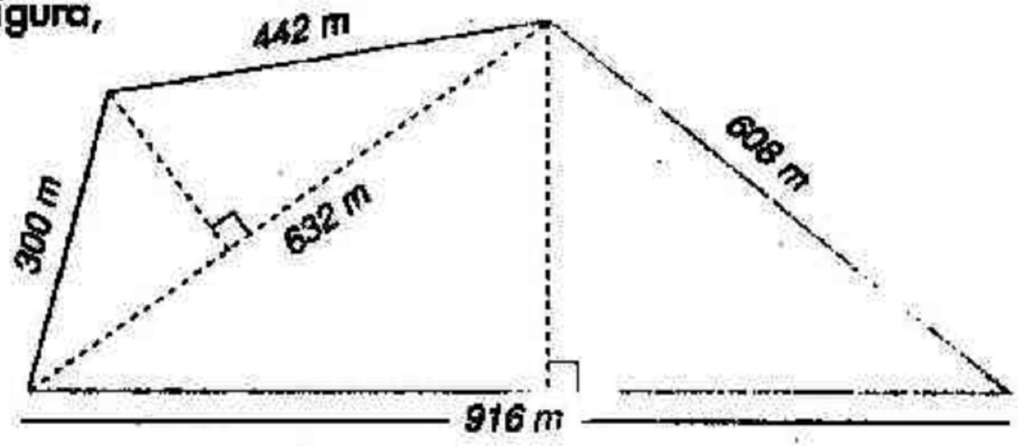
The « Chilian » solution

Decomposition of the figure and calculation after measuring on the figure

¿Puedes estimar el área de la parcela de la figura, a partir de las mediciones indicadas?

Solución: Podemos descomponer la parcela en pedazos triangulares como los indicados y reconstruir estos triángulos a partir de las mediciones tomadas.

¿Cómo calculamos ahora el área?



How can we compute the area now? Well, we determine the scale of the drawing, we measure the relevant altitudes and we obtain the area of each triangle (by multiplying each length of a base by the half of the corresponding altitude).

A surprising approach for French teachers

The « Chilian » Model

- The model is the drawing at a given scale.
- It allows mesuring relevant lengths on the drawing and obtain the corresponding lengths in reality.

A model more appropriate for French teachers

- Split into two triangles
- The problem is then to compute the area of a triangle whose lengths are known (without drawing and measuring)
- The solution is the « Heron formula »

$$p = \frac{a+b+c}{2} \qquad \mathbf{A} = \sqrt{p(p-a)(p-b)(p-c)}$$

- Students should prove the formula using
  - the Pithagorean theorem
  - algebraic calculation (a lot !)

Working Spaces

« Chilian »

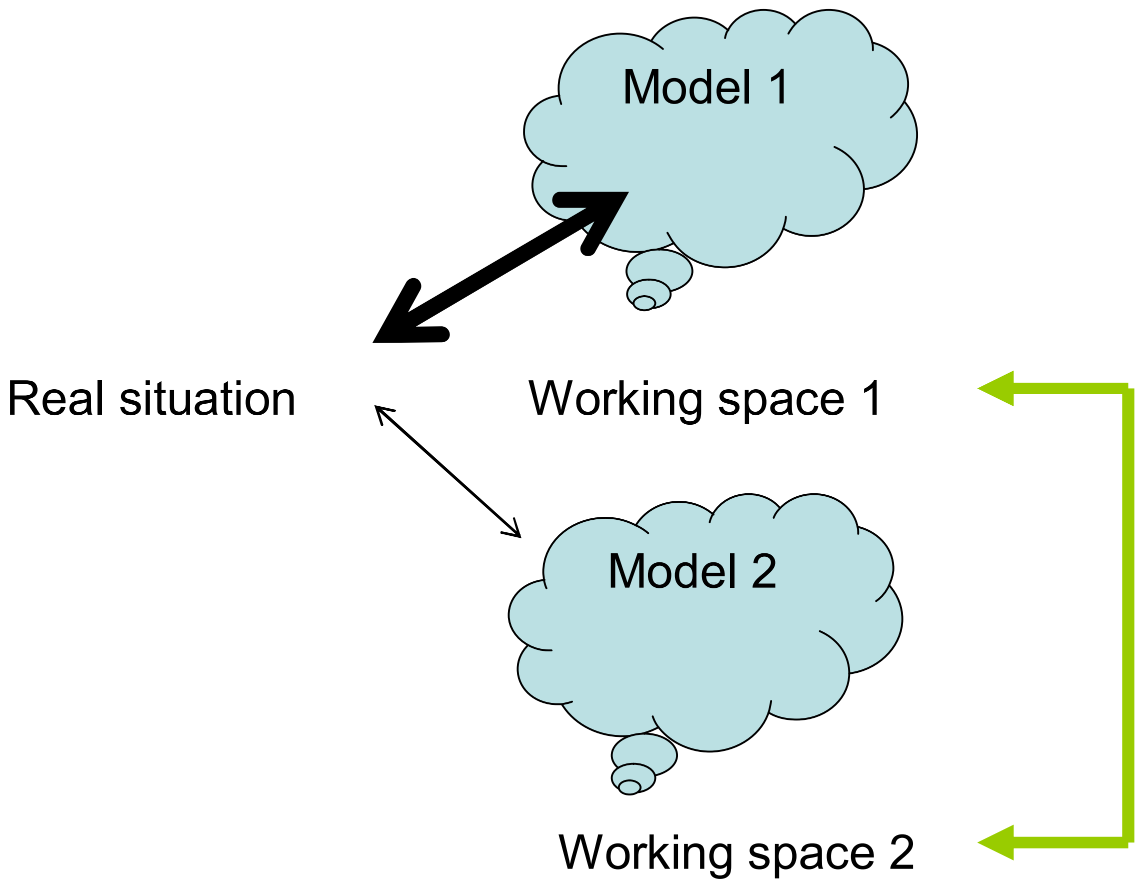
- Find out the scale
- Measure on the drawing (using a graduated ruler)
- Apply the scale to get the length of each altitude in reality
- Discuss the approximation

« French »

- No drawing, No measuring
- A lot of algebra
- Formal proof (theorems…)

- Working on a model is working in a specific mathematical space
- Each of the Working Spaces involved in a modelling process has its legitimacy and scope
- but they do not share
  - the means of action (instruments, symbols…)
  - the justifications of these actions
  - and the resulting conceptualizations.

Deeper understanding by connecting working spaces



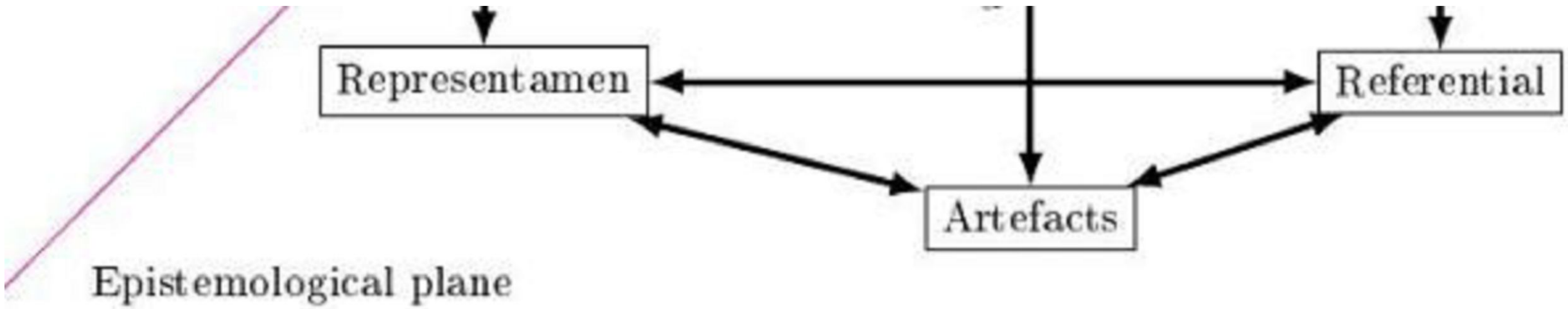
A Mathematical Working Space (MWS)

- An abstract space organized to ensure the mathematical work (in an educational context).
- Based on the articulation
  - An Epistemological level related to mathematical organization and posing the task
  - A Cognitive level related to individual activity and doing the task

The epistemologic level

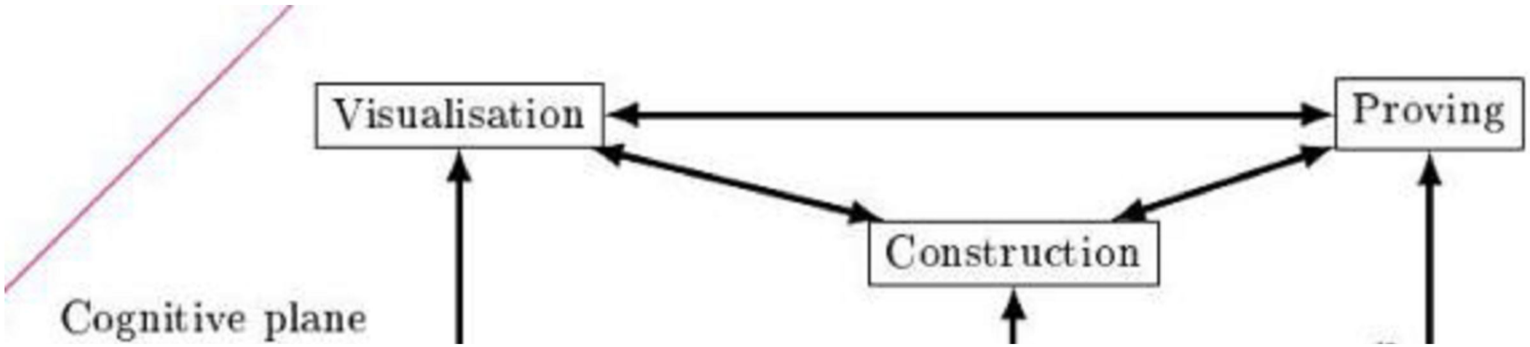
A network of three components :

- A Representamen (or sign) made of a set of tangible objects :
  - geometrical images, algebraic symbols, graphics,
  - or even concrete objects (tokens, mockup; photos ..)
- A set of artefacts such as material instruments or software or symbolic (techniques, algorithms…)
- A theoretical frame of reference based on definitions and properties.

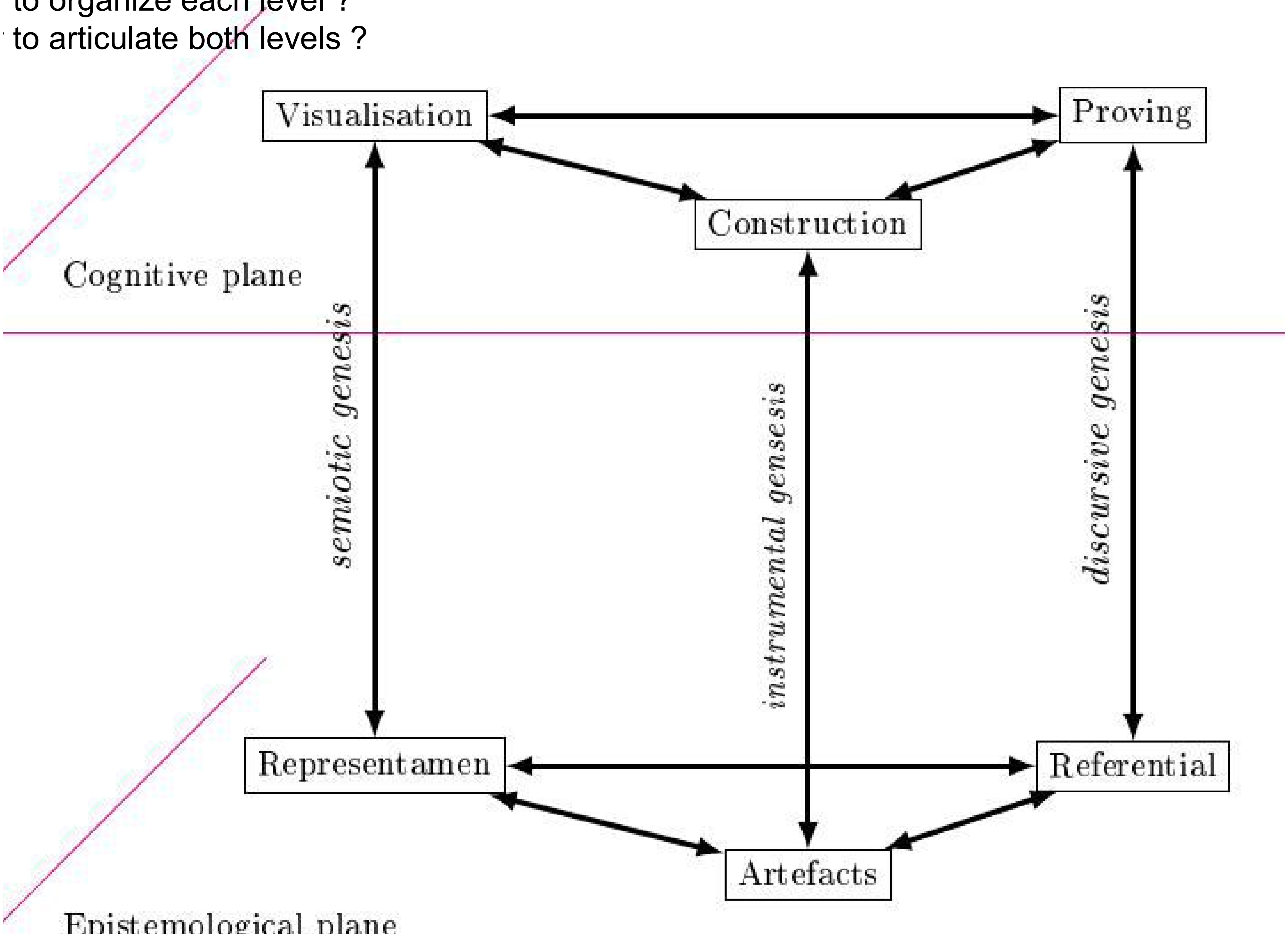


The cognitive level

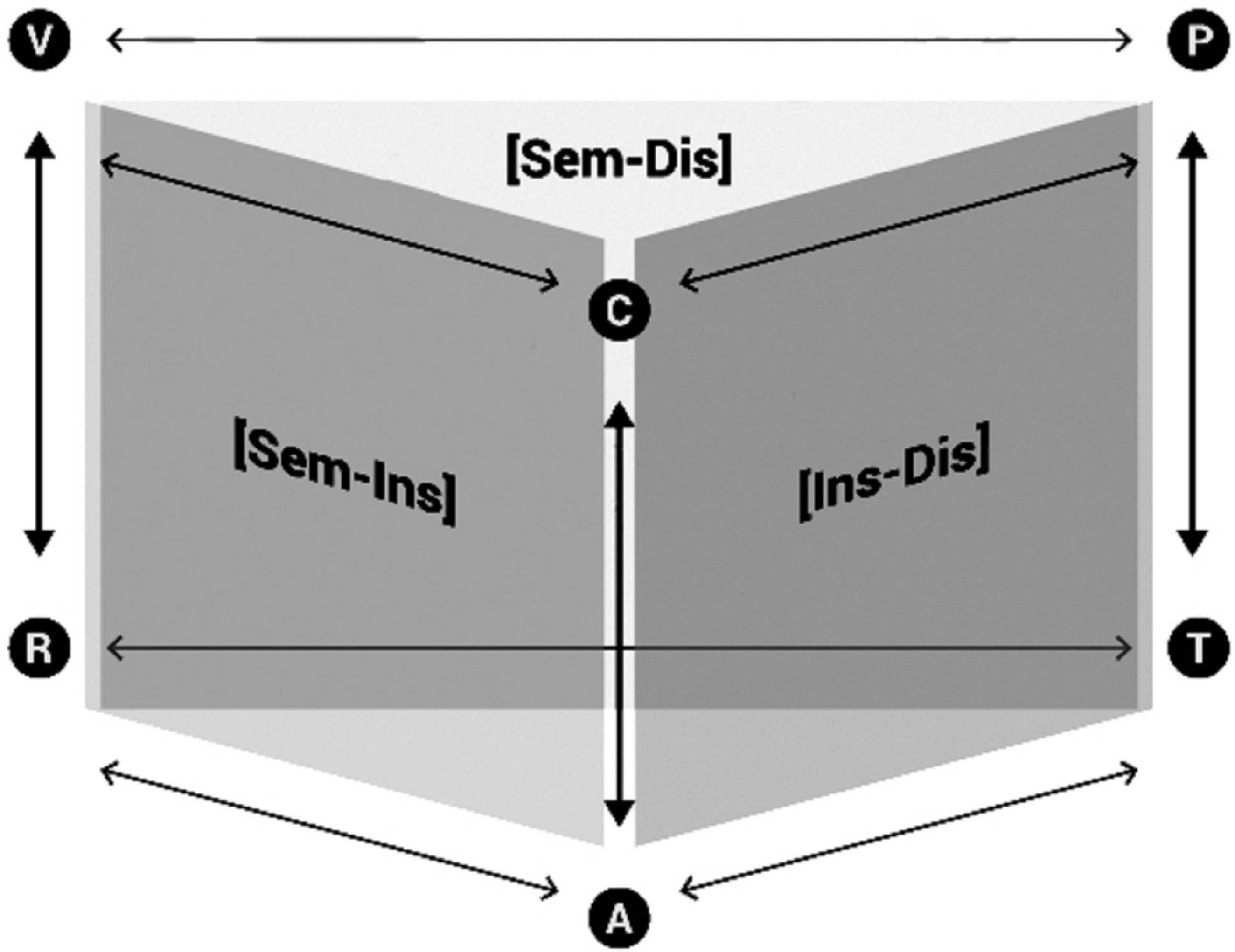
- A visualisation process connected to the representation of space and material support
- A construction process determined by the instruments (ruler, compass, etc.) and geometric configurations
- A discursive process which conveys reasoning and proof



- to organize each level ?
- to articulate both levels ?



Vertical Planes : Sem-Dis, Ins-Dis, Sem-Ins

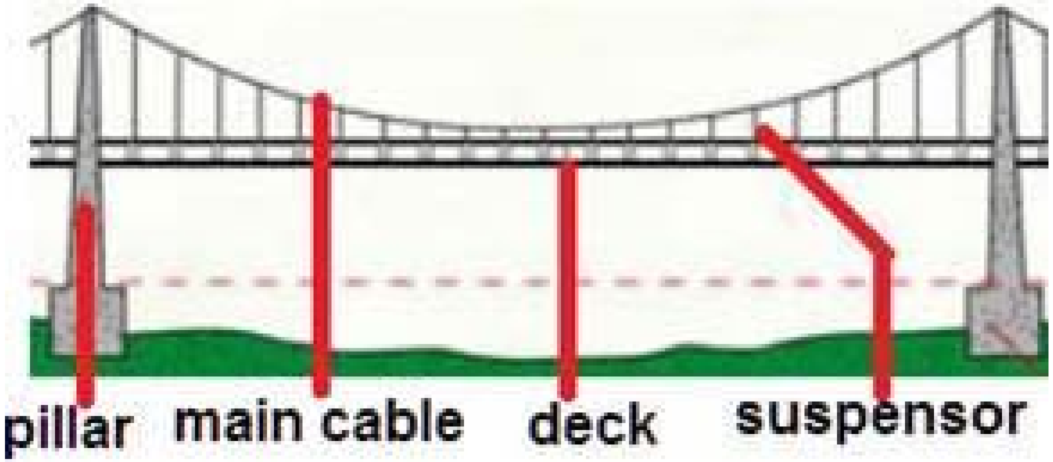


- How the idea of connected working spaces can help to
  - design
  - and evaluate

  modelling based teaching/learning situations?

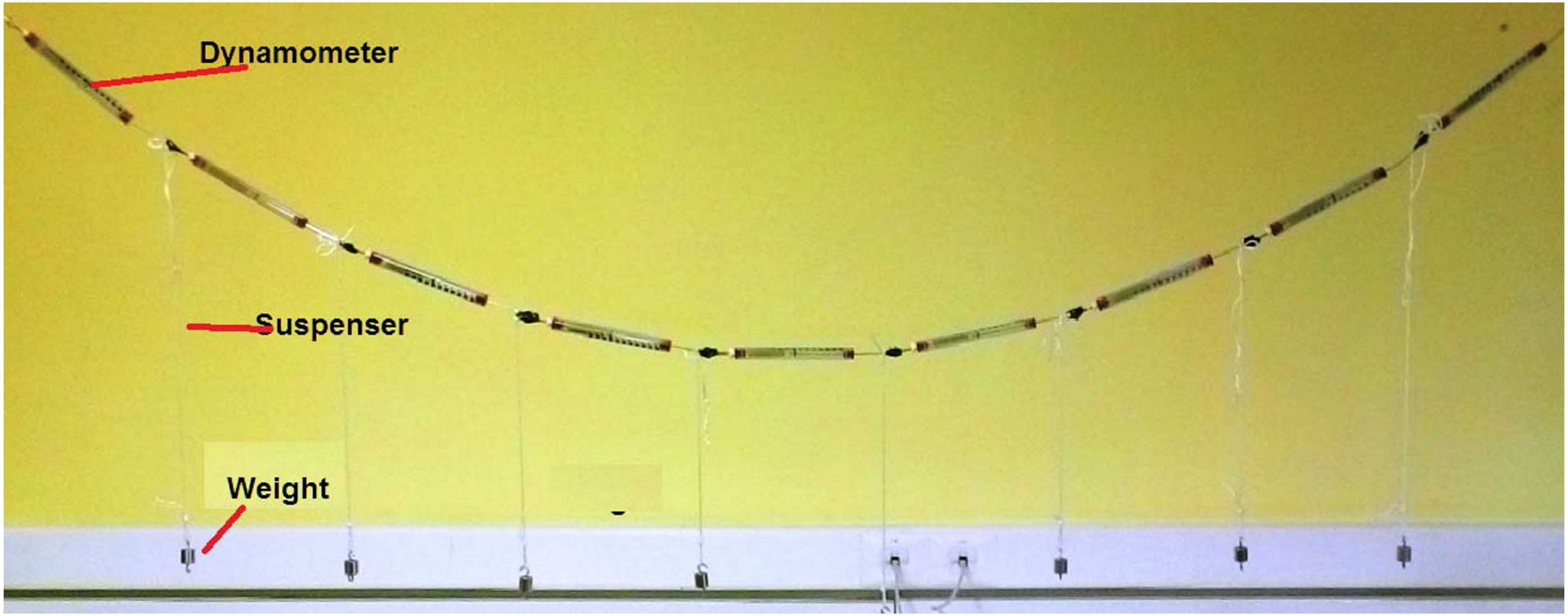
Modelling suspension bridges

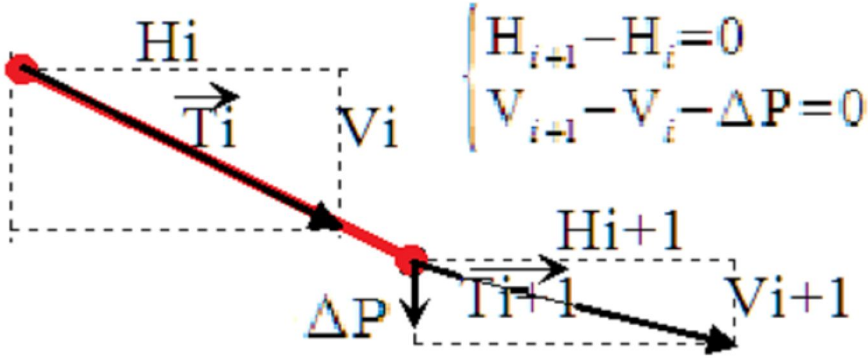
- Four models
- Four Working Spaces
- Classroom implementation (12th grade)
- Observation and evaluation
- Theses for discussion



- The deck is hung below main cables by vertical suspensors equally spaced.
- The weight of the deck applied via the suspensors results in a tension in the main cables.
- There is no compression in the deck and this allows a light construction and a long span (Golden Gate, Akashi kaikyō) .
- Not to be confused with
  - Catenary (deck follows the cable)
  - Straight cables (Rio-Antirio, Chalkis…)

A model as a static system

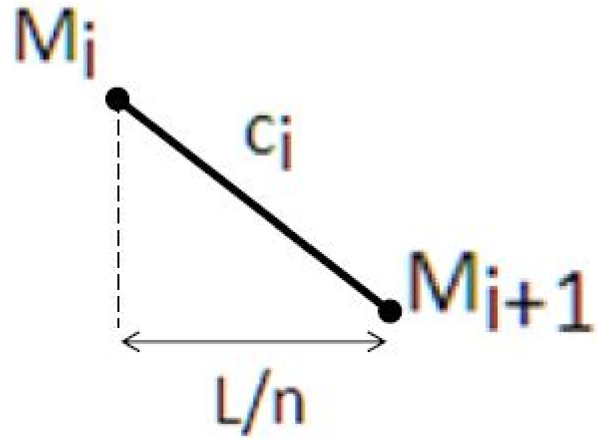




$$\begin{cases} H_{i+1} - H_i = 0 \\ V_{i+1} - V_i - \Delta P = 0 \end{cases}$$

A model in coordinate geometry

M_0 and M_n the anchoring points on the pillars,
and $M_1, M_2, \ldots, M_{n-1}$, the points where suspensors are attached on the cable, x_i, y_i the coordinates of M_i.
A model of the cable is the broken line $M_0 \ldots M_n$.



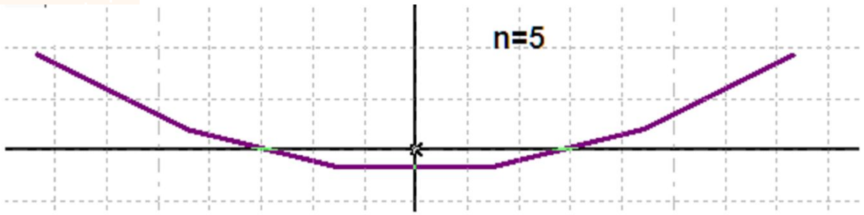
The slope c_i of a segment $[M_i, M_{i+1}]$ is the quotient of the vertical and horizontal component of the tension in this segment.

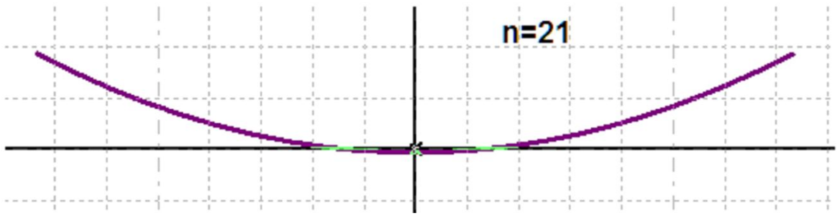
$$x_{i+1} = x_i + \frac{L}{n} \qquad y_{i+1} = y_i + \frac{L}{n} \cdot \frac{V_i}{H}$$

An algorithmic model

In the program below, the data comes from the golden gate bridge and the origin of the coordinate system is at the middle of the deck.
Weight of the deck: 20 MegaNewtons
Distance between two pillars: 1 280m
Elevation of pillars above the deck: 163m

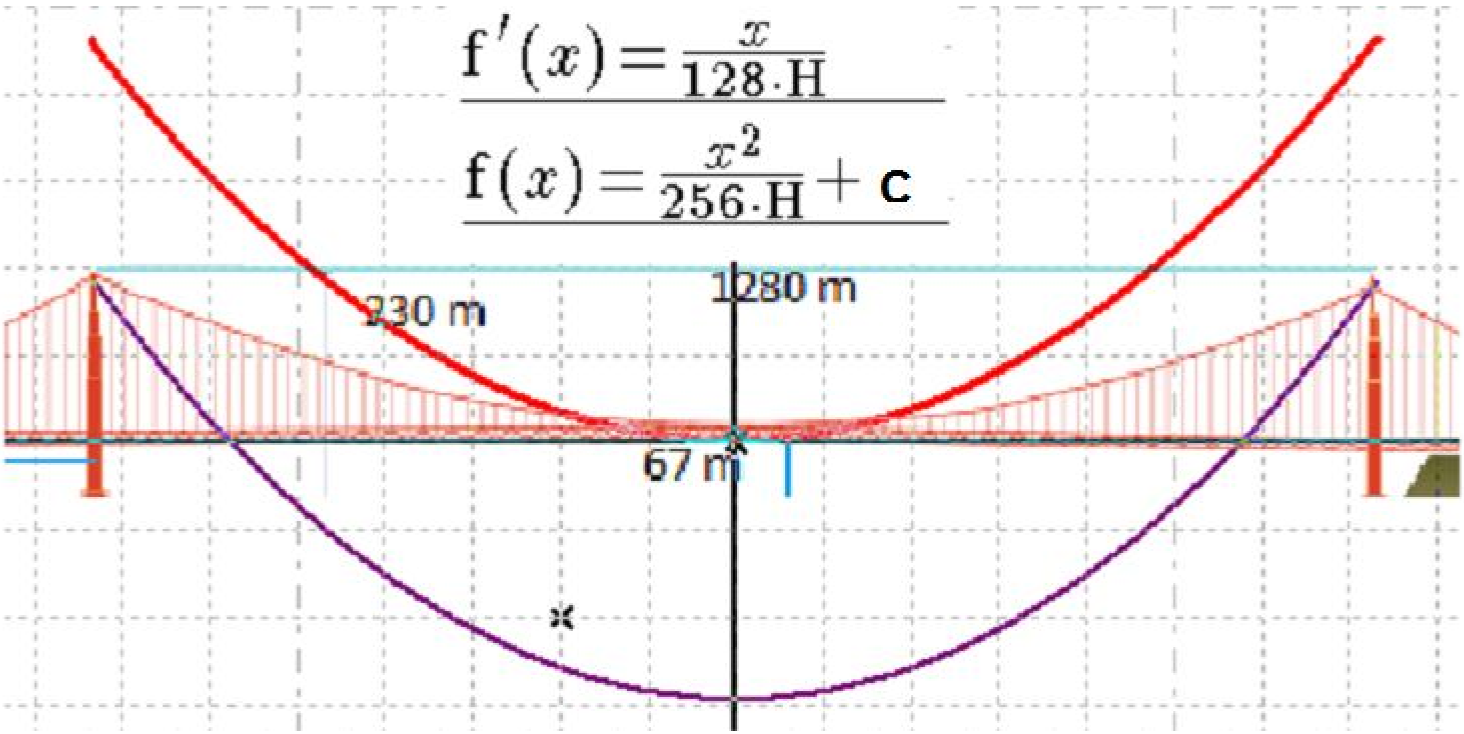
```
1://coordinates of the anchoring point
2:  x ← −640
3:  y ← 163
4://Tension on the anchoring point
5:  V ← −5
6:  Pour i allant de 1 à n
7:    x ← x + 1280/n
8:    y ← y + (1280/n)·(V/H)
9:    V ← V + 10/(n−1)
10: Fin Pour
```



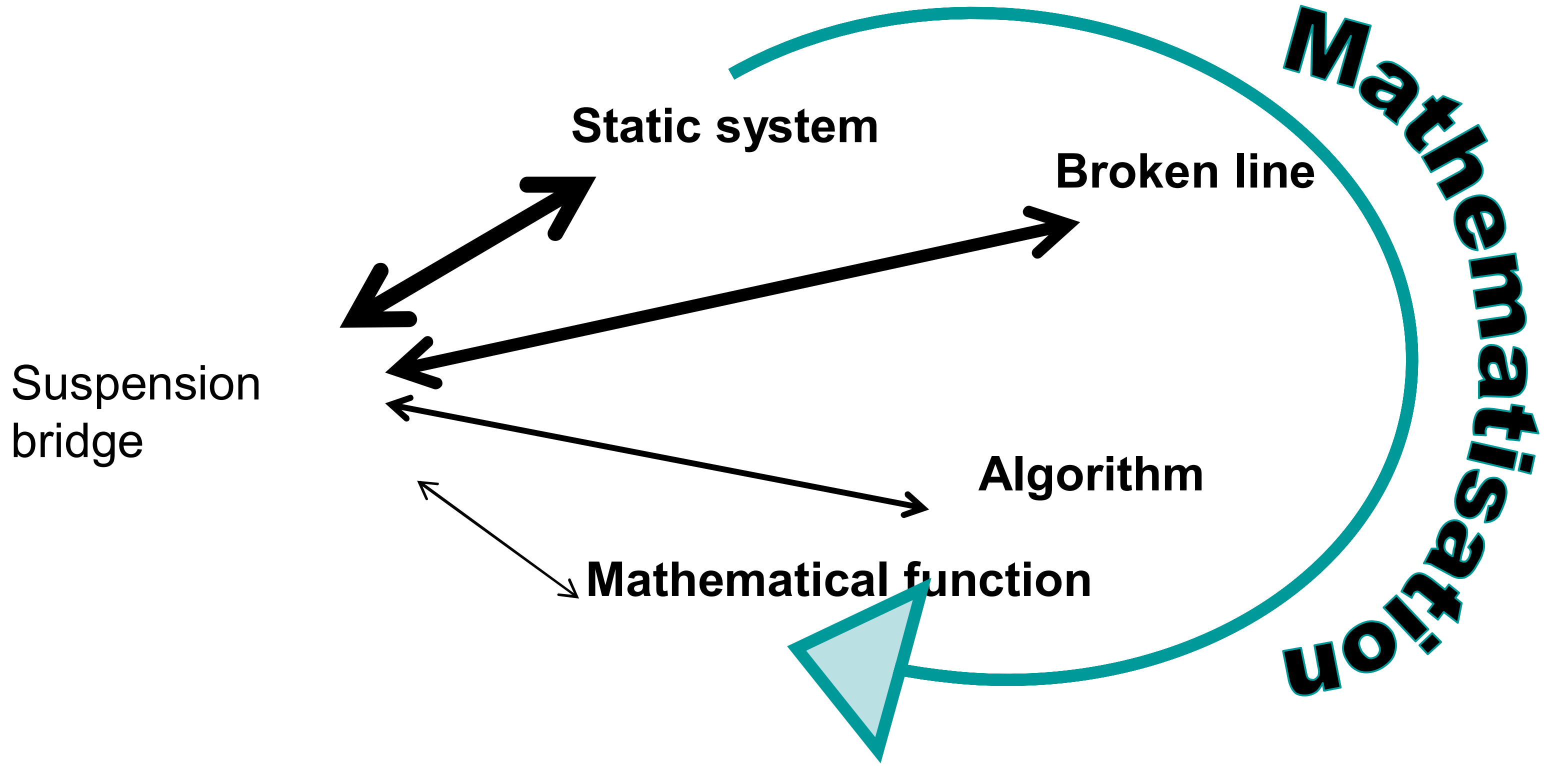


A continuous model, using a mathematical function

The horizontal component of the tension in the cable is a constant H.
The derivative V'(*x*) of the vertical component of the tension with regard to the horizontal position x, is also a constant P / 2L,
V(x)= P. *x* / 2L   f '(x)= V(x)=/H



All models are mathematical, some are more



Four working spaces

- **The Static systems working space**
  - Representamen: sequence of tensions at the connection points of the suspensors,
  - Rules: static equilibrium law and the properties of arithmetic progressions.
  - Artefacts: concrete measurement devices used in physics and mathematics, dynamometers, angle protractor.
- **The Geometrical working space**
  - Discrete model of a main cable as a sequence of points
  - The main rule: analytical definition of a segment
- **The algorithmic working space**
  - discrete model of a main cable as a continuous piecewise functions,
  - important artefact: programming environment
- **The mathematical functions space**
  - Continuous model of the cable as a standard function governed by classical rules in calculus.
  - symbolic capabilities (integration).
  - graphing
  - compare to a picture of the bridge and to the discrete model,
  - adjust the horizontal component H in order that the three models fit.

Classroom Implementation

- Modelling a suspension bridge implies interrelated concepts
  - in physics: tension, static equilibrium of forces,
  - In geometry and calculus: projection of vectors, slope of segments and gradient of curves, arithmetic progression and linear function, iterative algorithms, discrete and continuous models, limits and integration…
- The goal for students
  - not to "reinvent" each concept in isolation,
  - but rather to recognize how modelling involves understanding these concepts operationally and in interaction.

Classroom Implementation

- Phase 1
  - one hour long, prepared with the physics teacher.
  - Introduction to different kind of bridges and questions about suspension bridges,
  - Video illustrating the idea of tension along a horizontal rope
  - Practical experiment about horizontal and vertical components of tensions and static equilibrium of forces
- Phase 2
  - Groups of experts
  - Each group works on a model (A, B, C or D) from a task sheet
- Phase 3
  - Groups of discussion (mixed A, B, C and D)
  - Task for students: Find connections between models A, B, C, D in order to prepare a synthesis
- Phase 4
  - Whole class synthesis

The second phase

50 mn long, Students split into groups, each with a task

- Task A (static systems working space)
  Students have to study the sequence of horizontal and vertical components of tensions at the suspension points

- Task B (geometrical working space).
  Students have to compute the series of *x* and *y*-coordinates of the suspension points for a small number of suspensors.

- Task C (algorithmic working space).
  An algorithm given; they have to enter and execute the algorithm, interpret parameter n, and adjust parameter H

- Task D (mathematical functions working space).
  They have to search for a function *f* whose curve models the cable, find a formula for the derivative of f, then for f and adjust H

The third phase

50mn long. The students form new groups

Each new group made in order to bring together one or two students of each of the previous groups respectively doing task A, B, C and D.

Phase 2					
Gr A	Gr B	Gr C	Gr D		
A1	B1	C1	D1	Gr 1	Phase 3
A2	B2	C2	D2	Gr 2	
A3	B3	C3	D3	Gr 3	
A4	B4	C4	D4	Gr 4	

The third phase

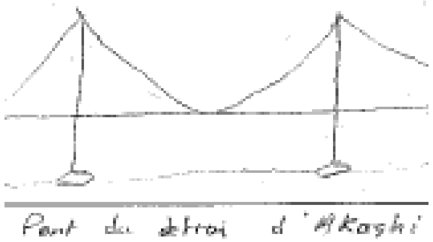
In each group

- share the findings
- write a report emphasizing the important points of the study.

Organization

- choosen in order that each student get a global understanding of the study of a problem, performing by himself some of the key tasks related to this problem
- consistent with the idea of several working spaces for the study of a problem

Observation: First phase

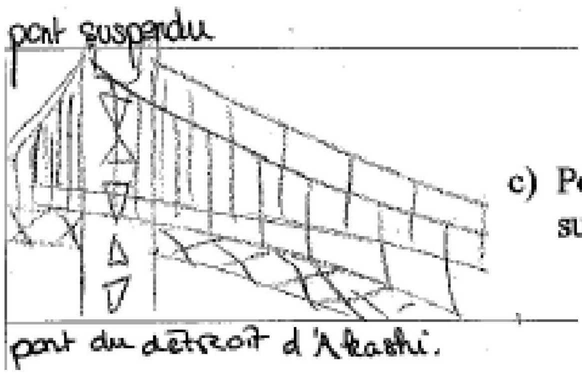


a) Pourquoi dans un pont suspendu les câbles porteurs ne sont -ils pas horizontaux ?

Le tablier appuit, et pousse le câble vers le bas (en tension)

c) Pour un pont suspendu, est-ce que la forme des câbles porteurs est due à la taille des suspentes ?

oui



a) Pourquoi dans un pont suspendu les câbles porteurs ne sont -ils pas horizontaux ?

Le tablier va pousser le cable vers le bas.

c) Pour un pont suspendu, est-ce que la forme des câbles porteurs est due à la taille des suspentes ?

Non, la forme est due aux forces exercées sur le cable.

Observation: Second phase

- Students doing task A (statics) mainly succeeded,
- Students doing task B (geometry)
  - Started by sketching a bridge with a lot of suspensors,
  - Took time to find the coordinates of the anchoring point, and had difficulties to use the formula given for the slope of the segments.
- Students doing task C (algorithmics)
  - Took time to enter the algorithm in Casyopée. small mistakes.
  - They could correct only when the observer helped them to analyze the algorithm.
  - Identified the parameter n as related to the number of suspensors.
  - After the observer explained that H is a tension, get aware that
  - increasing the value of this parameter "straightens" the cable.
- Students doing task D (mathematical functions)
  - found a formula for the vertical tension,
  - had difficulty to interpret the fact that the tension is in the direction of the tangent to the curve.

Observation: Third phase

- The parameter H was identified by students as playing a role in each task.
- Difficulty in task D to find the direction of the tangent to the curve and then the derivative of the function overcome thanks to student A
  – you just integrate the quotient of V and H
  – $f'(x) = \Delta y/\Delta x = V(x)/H$.
- Unfinished task B,
- Details of the algorithm in task C and tension in the continuous model (task D) not discussed.

Interview of students

- Situation more complex than usually
  - "we had to connect a lot of different things"
  - "not used to mix physic and mathematics".
- Awareness of the structure of a bridge progressed:
  - role of the suspensors
  - link between a suspension bridge and an arched bridge with regard on how the deck is supported.
  - link between the apparatus with two weights and a suspension bridge "with two suspensors".
- Correctly interpreted the algorithm of task C, and connect the evolution of H, and *x* and *y* respectively to task A and B.

Interview of students

- Difficulties in computing the coordinates in task B.
- No clear awareness of the function as a limit of a continuous piecewise function.
  - From graphical evidence they thought that it was more or less the same function for big values of n.
- The observer asked to explain why the gradient in a point of the curve is the quotient of V and H. The expected answer was that the tension has the direction of the tangent, but the students simply wrote $f'(x) = \Delta y/\Delta x = V(x)/H$ without more explanation.

Connections

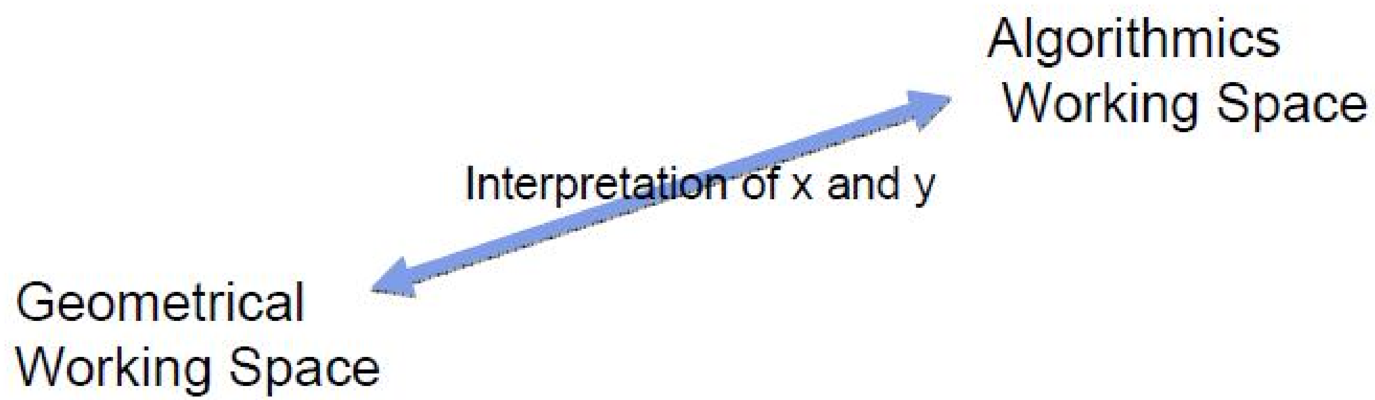
Statics Working Space ⟷ Algorithmics Working Space

Animation of parameters

n is the number of suspensors
H is the horizontal component of the tension along the cable

Semiotic-Instrumental genesis

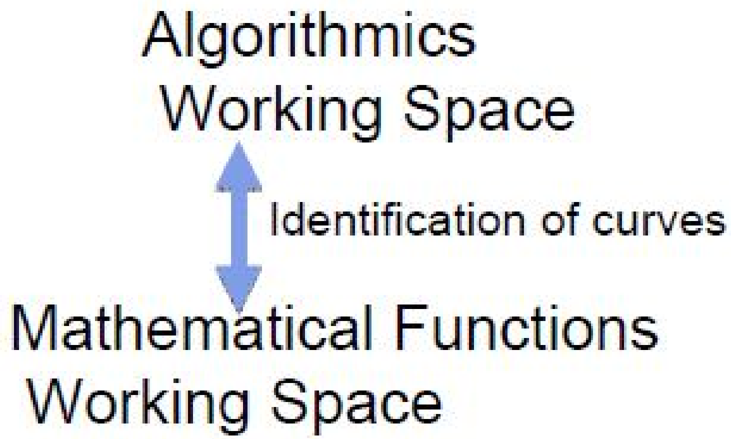
Connections



The evolution of x and y in the algorithm is consistant with the recurrence in the geometric model.

Instrumental-Discursive genesis

Connections



The curve of the mathematical function
is like in the discrete model for big values
of n

Semiotic Instrumental genesis

Connections

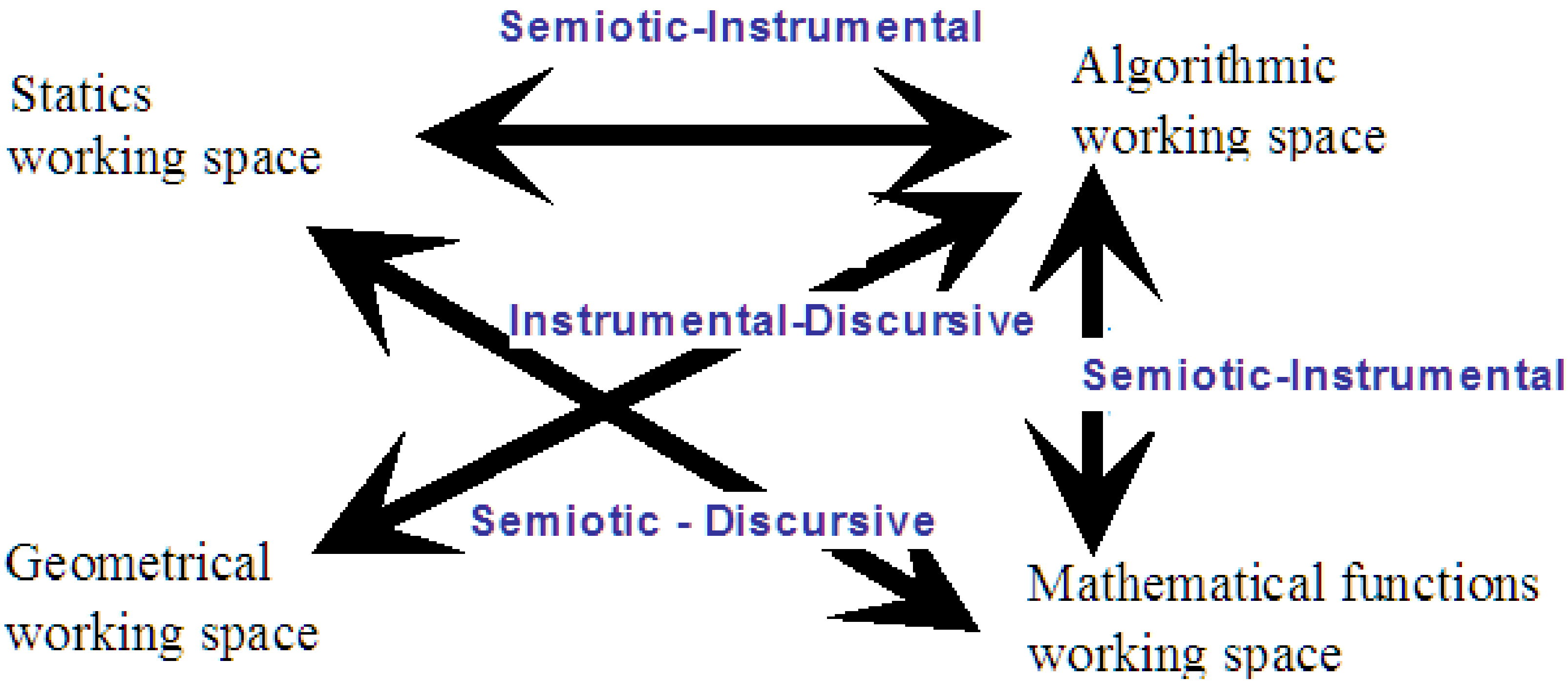
Statics
Working Space

Identification of
differential quotient
and dérivative

Mathematical Functions
Working Space

$$f'(x) = \Delta y/\Delta x = V(x)/H$$

Semiotic – Discursive genesis



- Geometrical does not connect much (as a difference with a « standard » modelling cycle)
- Algorithmic connects to the 3 others
- Variety of geneses through connections

Theses for discussion

- Modelling can be seen as active appropriation of models of varied natures for a given reality
- Activity on each model can be described as a work in a specific working space (objects, artefacts, rules)
- Understanding
  – can be thought of as an outcome from connections made by students between working spaces
  – can be described by specifying geneses associated to connections
- This helps design classroom situations to (re) mathematize complex objects in everyday life

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