Connected working spaces: the case of computer programming in mathematics education

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The goal of our talk is to present an extension of the framework of Mathematical working spaces that Alain Kunzniak created and that he discussed last tursday in the working group, and to illustrate by way of a particular case.

Concern: teaching/learning situations associating mathematical domains and other domains

Situations

- Co-variation and functions
  - Domains of sensual experience (e.g. dynamic geometry)
- Modelling
  - Everyday experience
  - Scientific or professional domains
- Programming
  - Computer science

- Lack of connection
  - between the experience in other domains and the mathematical formalism and techniques.
  - between the processes of solving and reasoning, in other domains and in mathematics
- Connections
  - thought of at epistemological level
  - But difficult to put into practice

When I am dealing with a theoretical framework, I like to ask what are the concerns that this framework is supposed to address and why. So here we are concerned with teaching/learning situations associating mathematical domains and other domains, and problems arising in these situations.

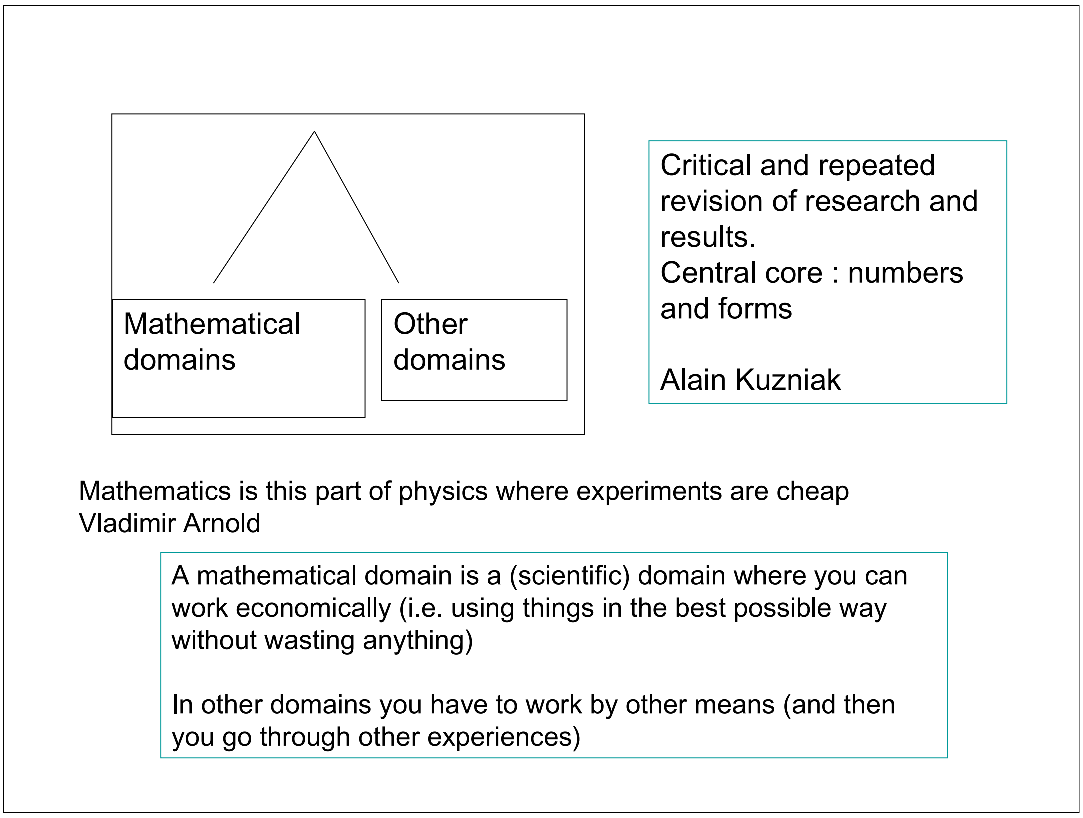
Three examples: Co-variation and functions. Many researchers stress the need to offer students domains of sensual experience of covaraition for instance by way of dynamic geometry before or in parallel with formal approaches of functions.

Modelling also associates domains of everyday experience or scientific or profressional domains in order that students make sense of mathematical notions and processes.

Last but not least, there is now a big emphasis in many curricula on the introduction of programming (or algorothmic, or coding) into mathematical activities.

The problems I see, looking at real classroom situations or even experimental situations is the lack of connection between the experience in the other domains and the mathematical formalism and techniques (this concerns for instance covariation and modelling). There is also a lack of connection between the processes of solving and reasoning in other domains on one side and in mathematics on the other side that concerns the three kinds of situations.

Especially with regard to programming, many authors write about connections with mathematics at epistemological level (for instance associating functions and variables in programming with their correspondents in mathematics, or stressing the role played by algorithms in mathematical domains for instance the euclid algorithm in number theory.



Remembering Thursday session of this working group, I think I have to explain what “other domains” are and how they differ from Matematical domains.

Alain characterized mathematical work as a critical and repeated revision of research and results. Certainly this is a preeminent feature in mathematics, but in other domains scientific or professional for instance, it is also done. Alain also said that mathematical work has a central core : numbers and forms.

As for me, I like the aphorism by Vladimir Arnold that Mathematics is this part of physics where experiments are cheap. This is not contradictory with Alain’s view, since it is cheaper, or say more economical to experiment on numbers and forms than for instance on sub-atomic particles.

So I propose this

A mathematical domain is a (scientific) domain where you can work economically (i.e.using things in the best possible way without wasting anything)

Other domains are domains where you have to work by other means (and then go through other experiences)

Current frameworks

Multirepresentation (*Duval, Arzarello…*)
Working on different representations of a math. entity.

Interplay between settings (*Douady*)
Changes of settings make students progress and their conceptions evolve.

Semiotic mediation (*Bartolini Bussi, Mariotti*)
1. Activities with artefacts
2. Production of signs (individual, collective)

Instrumental approach to use of artefacts in the math. activity (*many*)
1. Pragmatic and epistemic mediations
2. Interplay between knowledge about the artefact and knowledge in maths.

What are the current frameworks that could address concerns related to the situations I mentioned before ?

Often in these situations, a framework about multirepresentation is often. It is stressed that working on different representations of a mathematical entity helps students to make sense of this entity. Duval insisted on the conversion of registers, and when I look for instance at notations of the derivative in physics and mathematics, I see clearly the opportunity of this work of conversion. I indicate also Arzarello and his team because they worked a lot on representations in domains not directly mathematical, like motion.

A framework popular in France is Douady's settings. For Douady, a setting can be a mathematical domain like geometry or algebra, but also a domain of everyday life. She stresses that, in problem solving, changes of settings make students progress and their conceptions evolve.

Because artefacts are now most often present in "other domains" (they are not economical) I have to mention the framework of the Instrumental approach to use of artefacts. Many researchers including me dealt with this framework. Some insist on a distinction between Pragmatic and epistemic mediations by the artefact, meaning that a part of the work is directed towards the artefact and another towards mathematics. As for me, I studied the connections or interplay between the knowledge about the artefact and the knowledge in maths produced while using the artefact.

Finally, also related to artefacts, I have to mention the framework of semiotic mediation by Bartolini Bussi and Mariotti. They insist on the production of signs thanks to action with an artefact, and collective reflexion on these signs in what they call a didactical cycle.

A semiotic dimension

Multirepresentation (*Duval, Arzarello…*)
Working on different representations of a math. entity.

Interplay between settings (*Douady*)
Changes of settings make pupils progress and their conceptions evolve.

Semiotic mediation (*Bartolini Bussi, Mariotti*)
1. Activities with artefacts
2. Production of signs (individual, collective)

Instrumental approach to mathematical tools (*many*)
1. Pragmatic and epistemic mediations
2. Interplay between knowledge about the artefact and knowledge in maths.

If I try to classify these frameworks, I can see first a semiotic dimensions, related to signs, their production and their transformations.

An instrumental dimension

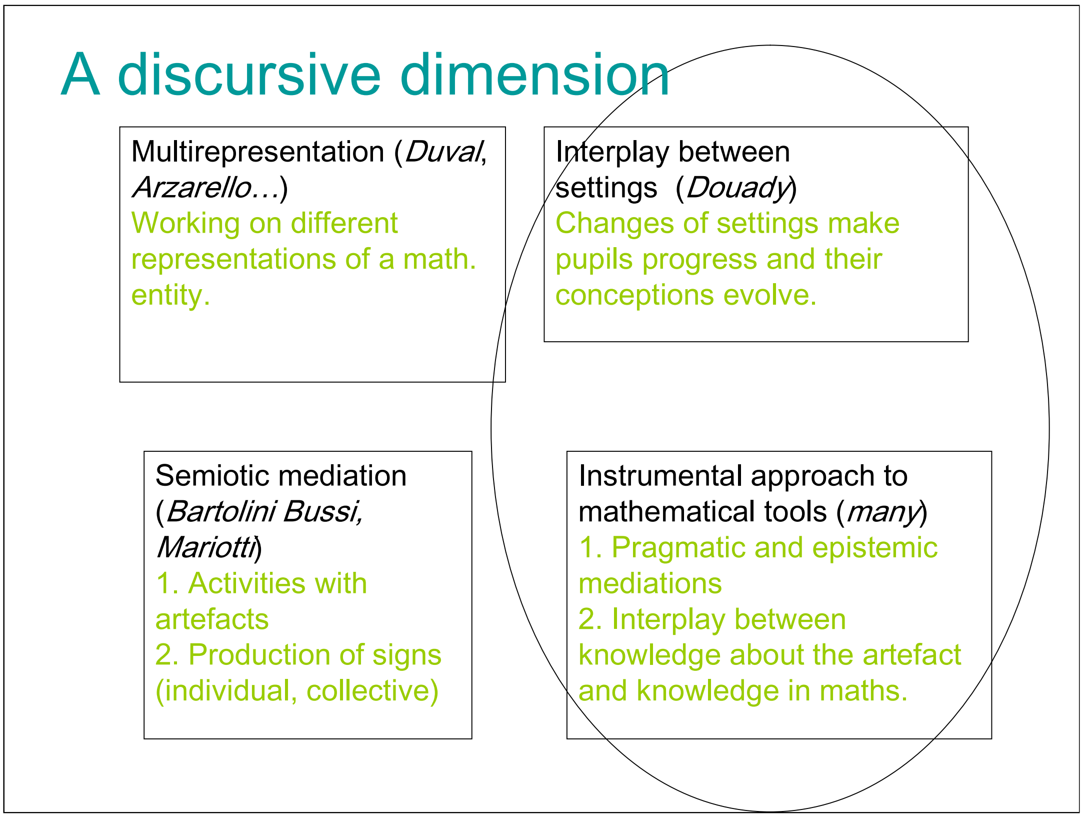
Multirepresentation (*Duval, Arzarello…*)
Working on different representations of a math. entity.

Interplay between settings (*Douady*)
Changes of settings make pupils progress and their conceptions evolve.

Semiotic mediation (*Bartolini Bussi, Mariotti*)
1. Activities with artefacts
2. Production of signs (individual, collective)

Instrumental approach to mathematical tools (*many*)
1. Pragmatic and epistemic mediations
2. Interplay between knowledge about the artefact and knowledge in maths.

I can see also an instrumental dimension linked to the of artefacts.



An finally a dimension linked to reasoning, argumenting, proving. I name this the discursive dimension to be consistent with the framework of Mathematical working spaces.

For a given situation on a given topic

A working space in another domain

- Semiotic dimension
- Instrumental dimension
- Discursive dimension

A working space in a mathematical domain

- Semiotic dimension
- Instrumental dimension
- Discursive dimension

So this is what we want to do. We want to build and analyse a situation on a given topic involving a mathematical and another domain.
We look at the three dimensions in the corresponding working spaces trying to contrast these and to see the possibilities of connection. Finally we analyse the connections and the dimensions implied by these connections in a teaching/learning situation both a priori and a posteriori.

Connecting Algorithmic and Mathematical Working Spaces

- A topic (upper secondary level, France)
  - Equation *f*(x)=0 for *f* defined on an interval [a, b]
- From a mathematical point of view,
  - the Intermediate Value Theorem (IVT) guarantees the existence of a solution in the interval [a;b] under the sufficient conditions: *f* is continuous and f(a) x f(b) ≤ 0 .
- From a computer programming point of view
  - algorithms able to find, for an arbitrary precision e, an interval with the property P(e): $|u - v| < e$ and f(u) x f(v) ≤ 0 e.g. scanning, dichotomy.
- Connections
  - With the sufficient conditions, an interval with the property P(e) contains a solution.
  - Sequences generated by an algorithm play a crucial role in the proof of TVI.

The topic we choose as an example is the solution of *f*(x)=0 for a function *f* defined on an interval [a, b]

We first look at this topic from a mathematical point of view,

the Intermediate Value Theorem (IVT) guarantees the existence of a solution in the interval [a;b] under the sufficient conditions: *f* is continuous and f(a) x f(b) ≤ 0 .

As a difference computer programming focuses on algorithms able to find, for an arbitrary precision e, an interval with the property P(e): $|u - v| < e$ and f(u) x f(v) ≤ 0 examples of algorithms are scanning and dichotomy.

There are also connections:With the sufficient conditions, an interval with the property P(e) contains a solution.

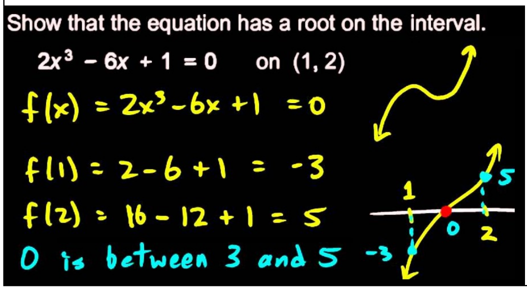
Sequences generated by an algorithm play a crucial role in the proof of TVI.

Mathematical Working Space

- Discursive dimension:
  - Focus on a mathematical solution.
  - Properties of functions
- Semiotic dimension
  - Usual mathematical formalism,
  - Formalism of infinite sequence.
  - Graphs (iconic)
- Instrumental dimension
  - Paper and pencil calculations,
  - Graphs.

Algorithmic Working Space

- Discursive dimension
  - Focus on the process of approximation
    - Termination, Effectivity, Efficiency
- Semiotic dimension
  - Specific markers of treatments
  - Variables
  - Mathematical expressions
- Instrumental dimension,
  - Execution by automatic device



$u \leftarrow a$
$v \leftarrow b$
While $v-u>e$
  $m \leftarrow \frac{u+v}{2}$
  If $f(u)\cdot f(m)>0$ then
  $u \leftarrow m$
Else
  $v \leftarrow m$
Endif
End While
Return $[u;v]$

Here we analyse the two working spaces in the three dimensions.

I just mentionned the two different focuses in mathematics and programming. The consequence is two different discursive dimension, one centred on the properties of functions that guarantee a solution, and the other on properties of the algorithms like Termination, Effectivity, Efficiency

The semiotic dimensions are also different. The usual mathematical formlism including sequences involved in a proff of the IVT on one side, Specific markers of treatments while, if then, varaibles whose statute is not so consistent with the idea of variables in mathematics

With regard to the Instruments, Paper and pencil is obvious as an instrument in mathematics, but graphs of functions can be used also as we shall see.

The Instrumental dimension in the Algorithmic Working Space is in the Execution by an automatic device that gives sense to the text of the algorithm by producing results for a variety of entries.

Execution by automatic device

A Situation

Objective: students become aware of conditions for a solution, independently of the effectivity of the algorithm

f function defined on $(1\ ;\ 2)$

$f(x) = \frac{1}{x^2-2}$ "hidden"

Task : enter and execute this algorithm for the function for various values of the threshold e.

Does the interval returned by the algorithm actually contain the solution?

Look at the graph of function and discuss.

$u \leftarrow a$
$v \leftarrow b$
While $v-u>e$
  $m \leftarrow \frac{u+v}{2}$
  If $f(u)\cdot f(m)>0$ then
    $u \leftarrow m$
  Else
    $v \leftarrow m$
  Endif
End While
Return $[u;v]$

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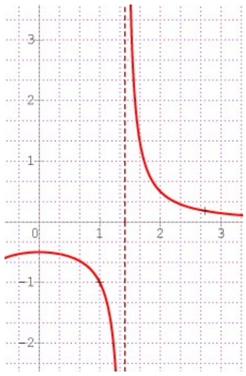
As indicated in the paper, we organised a series of situations at the three levels of French upper secondary levels with various relationship between the two working spaces. I will here report on one situation. The situation dealt with the dichotomy algorithm witch is effective for every function changing its sign over the interval in the sense that it will return an intervall with the condition P(e) We wanted that students become aware that it does not guarantee that the interval returned by the algorithm approximates a solution.

So we introduced this function in the system, the formula was hidden. So the algorithm referred to the function by its name, and the students could graph the function, but they ahd not the formula.

A situation

A conflict between effectivity in the AWS and existence of a solution in the MWS (discursive dimensions)



Console

```
***Algorithme lancé***
Entrer a : 1
Entrer b : 2
Entrer e : 0.000000001
[1 ; 1.5]
[1.25 ; 1.5]
[1.375 ; 1.5]
[1.375 ; 1.4375]
[1.40625 ; 1.4375]
[1.40625 ; 1.421875]
[1.4140625 ; 1.421875]
[1.4140625 ; 1.4179688]
[1.4140625 ; 1.4160156]
[1.4140625 ; 1.4150391]
[1.4140625 ; 1.4145508]
```

solved thanks to work
•recognizing properties of function on graphs (semiotic/discursive dimensions in the MWS)
•coordinating the instrumental dimensions in the AWS and in the MWS

instrumental
AWS (execution) ⟷ MWS (graph)
discursive
AWS (effectivity) ⟷ MWS (existence)

So what we expected was a conflict between effectivity in the AWS and existence of a solution in the MWS that concerns the discursive dimensions in both working spaces

the students actually found after the execution that for increasing values of the theshold the intervals become closer and should then approximate a solution.

Looking at the graph, they found it unusual as compared to continuous functions they were used to, and most suspected something. To better appreciate they adapted the algorithm to produce the values of the varaibles at every step of the iteration, and they reported these values on the graph of the function and found that the values of the function were increasing, instead of decreasing towards zero. So they became aware that, when using the algorithm with other functions before, these functions had a special property that allowed the algorithm to actually return approximations of a solution.

So there is no definitive conclusion, this is an ongoing research, but we think that the framework helped to better characterize the wotk in the two domains and the potential and actual connections between the domains.